

Perturbation of self-similar sets and some regular configurations and comparison of fractals

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Abstract

We consider several distances between two sets of points, which are modifications of the Hausdorff metric, and apply them to describe some fractals such as δ -quasi-self-similar sets, and some other geometric notions in Euclidean space, such as tilings with quasi-prototiles and patterns with quasi-motifs. For the δ -quasi-self-similar sets satisfying the open set condition we obtain the same result as a classical theorem due to P. A. P. Moran. In this paper we try to gaze on fractals in an aspect of their “form” and suggest a few of related questions. Finally, we attempt to inquire an issue — what nature and behavior do non-crystalline solids that approximate to crystals show?

1 Introduction

Generally fractals are considered to possess three important features: *form*, *chance* and *dimension*, just as indicated in the titles of [71] and [72]. The dimension has been a very important and fundamental subject in researches of fractals (see e.g. [35] and [73]). Random fractals as more natural description of things in nature are also extensively investigated by experts in many subjects (see e.g. [8], [29], [35], [46], [54], [73] and [79]). The research of fractals is also closely connected to geometric measure theory (refer to [28], [37], [77] and [92], etc) and other scientific subjects (refer to [36], [72], [73], [75] and [76], etc).

Usually fractals are also considered to possess recursive or recurrent structure. Self-similarity is one of simple and important natures of fractals, where the self-similarity may often be approximate or statistical.

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A mathematical (strict) self-similar set as an extension of the classical Cantor set has been investigated deeply and extensively (see [35], [53] and [87], etc). In this paper we will describe an approximate self-similar set in a quantitative respect, considering it as a perturbation of a strict self-similar set.

However, for the fine structure of a (mathematical) self-similar set, any small perturbation will probably destroy its fractal details. For example, assume that F is a self-similar set (fractal) in \mathbb{R}^n (n -dimensional Euclidean space) satisfying

$$h(F, E) \leq \varepsilon,$$

where $\varepsilon > 0$, E is a subset of \mathbb{R}^n and h is the Hausdorff metric. Then no matter how small ε is, E can be chosen as a usual Euclidean figure.

The research on tilings, patterns and packings has a long history, which was once advanced by Hilbert's 18th problem and developments of other subjects, especially crystallography. For the context and introduction, let us refer to [19], [23], [44], [47], [48], [49], [50], [51] and [97], etc.

In this paper, first we consider several modified Hausdorff metrics, which are called *shape differences* and are proved to be complete metrics (in appropriate spaces)(see Section 2), and then we perturb self-similar sets scale by scale using them. We also get the Hausdorff dimension of the perturbed self-similar set by a classical method ([53] and [87]) (Section 3). Furthermore, we approach the form of fractals by comparison and suggest the notion of *splines of fractals*, the *fractal index* and the *similarity index* to indicate inside structures and complexity of fractals (Section 4). In the last part (Section 5), we try to modify some classical concepts in tiling, pattern, packing and crystallography to lead valuable investigations from other people. As an example we extend a basic result about engulfing in the research of patterns (see [50, 5.1.1]). In the paper we suggest a few of related questions for consideration.

In fact, the modified Hausdorff metric has been studied by experts in computational geometry for quite a long time (see e.g. [1], [18], [22], [56], etc). But according to my limited knowledge the experts mainly make investigations in the aspects of algorithms and their time and so on. A modified Hausdorff metric (the Hausdorff-Chabauty distance) has been also defined in [106] and some wonderful observations about the Mandelbrot set and Julia sets were proved there.

The introduction to researches on fractals (in many subjects of science), tilings, patterns, packings and crystals can not be included here for their tremendous amount. For example, only in the investigation of the Mandelbrot and Julia sets the researches have been so plentiful that perhaps one ordinary book can not include them all. Here we just mention one result (about “topological form” of a fractal) that the Mandelbrot set of $f_c(z) = z^2 + c$ is connected, which was observed by B. B. Mandelbrot (see [73] and [74]) and proved by A. Douady and J. Hubbard (see [27]).

In this paper by the term “fractal” we mean not only an irregular object but also sometimes a regular one.

In the preface of [97] M. Senechal said: “... Some of these tools were new to me, and although I have enjoyed the adventure of learning how to use them, I am also aware that I may have made errors or am ignorant of the relevant literature. I will be grateful for any criticisms, comments, and suggestions: the adventure continues.” This is also my feeling while writing this paper. It is my hope that this paper might play a role of “casting a brick to attract a gem”.

2 Shape differences

Let \mathbf{X} be a complete metric space with a metric d . Define

$$d(A, x) = d(x, A) := \inf\{d(x, a) : a \in A\}$$

for \mathbf{X} and a subset A of \mathbf{X} . For $\delta \geq 0$ we denote

$$\mathcal{P}_\delta(A) = \mathcal{P}(A, \delta) := \{x : d(x, A) \leq \delta\},$$

which is called a δ -parallel body of A , and

$$\mathcal{N}_\delta(A) = \mathcal{N}(A, \delta) := \{x : d(x, A) < \delta\}$$

is called a δ -neighborhood of A or an open δ -parallel body of A .

Suppose A and B are two nonempty subsets of \mathbf{X} . Define

$$h(A, B) := \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\}, \quad (2.1)$$

which is called the *Hausdorff distance (metric)* between A and B . It also follows that

$$h(A, B) = \inf \{ \delta : A \subseteq \mathcal{P}_\delta(B), B \subseteq \mathcal{P}_\delta(A) \}. \quad (2.2)$$

Let

$$\mathcal{C}(\mathbf{X}) := \{C : C \text{ is a nonempty closed bounded subset of } \mathbf{X}\}.$$

Then $(\mathcal{C}(\mathbf{X}), h)$ is a complete metric space (see [37, 2.10.21] and [92, § 2.6]). Here and hereafter \mathbb{R} indicates the real numbers and \mathbb{P} indicates the set of positive integers.

2.1. Shape differences. We consider n -dimensional Euclidean space \mathbb{R}^n with usual Euclidean distant d .

2.1.1. We say that two nonempty subsets A and B of \mathbb{R}^n are *isometrically equivalent* if there exists an isometry $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $B = \varphi(A)$, denoted by $A \sim B$. Obviously this relation is an equivalence relation. The equivalence class of A is denoted \tilde{A} .

From (2.1) or (2.2) it follows easily that

2.1.2. Lemma. *Let φ be an isometry in \mathbb{R}^n and let A and B be nonempty subsets of \mathbb{R}^n . Then $h(\varphi(A), \varphi(B)) = h(A, B)$. \square*

2.1.3. Definition. For nonempty subsets A and B of \mathbb{R}^n we define

$$\tilde{h}(A, B) := \inf \{h(\varphi(A), \psi(B)) : \varphi \text{ and } \psi \text{ are isometries in } \mathbb{R}^n\}.$$

By Lemma 2.1.2 we have the following expressions:

$$\begin{aligned} \tilde{h}(A, B) &:= \inf \{h(A_1, B_1) : A_1 \sim A, B_1 \sim B\} \\ &= \inf \{h(A_0, B_1) : B_1 \sim B\} \quad (\text{where } A_0 \sim A) \\ &= \inf \{h(A_1, B_0) : A_1 \sim A\} \quad (\text{where } B_0 \sim B). \end{aligned}$$

If $A_1 \sim A_2$ and $B_1 \sim B_2$, then $\tilde{h}(A_1, B_1) = \tilde{h}(A_2, B_2)$. So we may define

$$\tilde{h}(\tilde{A}, \tilde{B}) := \tilde{h}(A, B).$$

We call $\tilde{h}(\tilde{A}, \tilde{B}) = \tilde{h}(A, B)$ the (*absolute*) *shape difference* between \tilde{A} and \tilde{B} or between A and B .

2.1.4. Denote

$$\tilde{\mathcal{C}}(\mathbb{R}^n) := \left\{ \tilde{C} : C \text{ is a nonempty compact subset of } \mathbb{R}^n \right\}.$$

Then $\tilde{\mathcal{C}}(\mathbb{R}^n) = \left\{ \tilde{C} : C \in \mathcal{C}(\mathbb{R}^n) \right\}$.

2.2. Theorem. $(\tilde{\mathcal{C}}(\mathbb{R}^n), \tilde{h})$ is a complete metric space.

Proof. At first we show that \tilde{h} is a distance function on $\tilde{\mathcal{C}}(\mathbb{R}^n)$. Let A, B and $C \in \tilde{\mathcal{C}}(\mathbb{R}^n)$.

(i) It is trivial that $\tilde{h}(\tilde{A}, \tilde{B}) = \tilde{h}(\tilde{B}, \tilde{A})$.

(ii) The triangle inequality $\tilde{h}(\tilde{A}, \tilde{B}) \leq \tilde{h}(\tilde{A}, \tilde{C}) + \tilde{h}(\tilde{C}, \tilde{B})$ follows from that for any $\varepsilon > 0$ we have

$$\begin{aligned} \tilde{h}(\tilde{A}, \tilde{B}) &\leq h(A, B_1) \leq h(A, C_0) + h(C_0, B_1) \\ &< \left(\tilde{h}(\tilde{A}, \tilde{C}) + \frac{\varepsilon}{2} \right) + \left(\tilde{h}(\tilde{C}, \tilde{B}) + \frac{\varepsilon}{2} \right) \\ &= \tilde{h}(\tilde{A}, \tilde{C}) + \tilde{h}(\tilde{C}, \tilde{B}) + \varepsilon, \end{aligned}$$

where $C_0 \sim C$ is chosen to satisfy

$$h(A, C_0) < \tilde{h}(\tilde{A}, \tilde{C}) + \frac{\varepsilon}{2}$$

and then $B_1 \sim B$ is chosen such that

$$h(C_0, B_1) < \tilde{h}(\tilde{C}, \tilde{B}) + \frac{\varepsilon}{2}.$$

(iii) Obviously $\tilde{h}(\tilde{A}, \tilde{B}) \geq 0$ and if $\tilde{A} = \tilde{B}$ then $\tilde{h}(\tilde{A}, \tilde{B}) = 0$. Below we will prove that if $\tilde{h}(\tilde{A}, \tilde{B}) = 0$ then $\tilde{A} = \tilde{B}$.

2.2.1. Definition. Denote the set of $l \times m$ matrices with entries in \mathbb{R} by $\mathbb{R}^{l \times m}$ ($l, m \in \mathbb{P}$). If $P = (p_{ij})_{l \times m} \in \mathbb{R}^{l \times m}$, then define

$$\|P\| := \sum_{i=1}^l \sum_{j=1}^m |p_{ij}|.$$

2.2.2. Lemma and Definition. Let $P_k = (p_{ij}^{(k)})_{l \times m}$ ($k \in \mathbb{P}$), $P = (p_{ij})_{l \times m} \in \mathbb{R}^{l \times m}$. Then

$$p_{ij}^{(k)} \rightarrow p_{ij} \quad (k \rightarrow +\infty)$$

(for $i = 1, \dots, l$ and $j = 1, \dots, m$) if and only if

$$\|P_k - P\| \rightarrow 0 \quad (k \rightarrow +\infty),$$

where P is unique. We say that the sequence $\{P_k\}$ of matrices has the *limit* P or $\{P_k\}$ *approaches* P , denoted by $\lim_{k \rightarrow +\infty} P_k = P$ or $P_k \rightarrow P$ ($k \rightarrow +\infty$).

2.2.3. Lemma. (1) If $P, Q \in \mathbb{R}^{l \times m}$, then

$$\|P + Q\| \leq \|P\| + \|Q\|.$$

(2) If $P = (p_{ij})_{l \times m} \in \mathbb{R}^{l \times m}$ and $Q = (q_{\alpha\beta})_{m \times s} \in \mathbb{R}^{m \times s}$, then

$$\|PQ\| \leq \|P\| \|Q\|.$$

Proof. (1) is obvious and (2) follows from

$$\begin{aligned}
\sum_{i=1}^l \sum_{\beta=1}^s \left| \sum_{j=1}^m p_{ij} q_{j\beta} \right| &\leq \sum_{i=1}^l \sum_{\beta=1}^s \sum_{j=1}^m |p_{ij}| |q_{j\beta}| \\
&\leq \sum_{i=1}^l \sum_{j=1}^m \sum_{\alpha=1}^m \sum_{\beta=1}^s |p_{ij}| |q_{\alpha\beta}| \\
&= \left(\sum_{i=1}^l \sum_{j=1}^m |p_{ij}| \right) \left(\sum_{\alpha=1}^m \sum_{\beta=1}^s |q_{\alpha\beta}| \right).
\end{aligned}$$

2.2.4. Lemma. Let P_k ($k \in \mathbb{P}$), $P \in \mathbb{R}^{l \times m}$. If $P_k \rightarrow P$ and $Q_k \rightarrow Q$, then $P_k Q_k \rightarrow PQ$.

Proof. From $Q_k \rightarrow Q$ and $\|Q_k\| \leq \|Q_k - Q\| + \|Q\|$ we know that $\|Q_k\|$ is bounded, i.e., there exists a constant $M > 0$ such that $\|Q_k\| \leq M$. By Lemma 2.2.3 we have

$$\begin{aligned}
\|P_k Q_k - PQ\| &= \|(P_k - P)Q_k + P(Q_k - Q)\| \\
&\leq \|P_k - P\| \|Q_k\| + \|P\| \|Q_k - Q\| \\
&\leq M \|P_k - P\| + \|P\| \|Q_k - Q\| \rightarrow 0,
\end{aligned}$$

which implies $P_k Q_k \rightarrow PQ$.

2.2.5. Lemma. Let $P_k = \left(p_{\alpha\beta}^{(k)} \right)_{l \times m} \in \mathbb{R}^{l \times m}$ and $\|P_k\| \leq M$, where $M > 0$ is a constant. Then there exists a subsequence $\{k_j\}_{j=1}^\infty$ of the natural number sequence and $P \in \mathbb{R}^{l \times m}$ such that $P_{k_j} \rightarrow P$ ($j \rightarrow +\infty$).

Proof. From $\|P_k\| \leq M$ we know that for all α and β ($\alpha = 1, \dots, l$; $\beta = 1, \dots, m$) the number sequences $\left\{ p_{\alpha\beta}^{(k)} \right\}_{k=1}^\infty$ are bounded. When $(\alpha, \beta) = (1, 1)$, we can get a number sequence $\left\{ p_{11}^{(k_j)} \right\}_{j=1}^\infty$, which has a limit p_{11} . Then consider $(\alpha, \beta) = (1, 2)$, i.e., a number sequence $\left\{ p_{12}^{(k_j)} \right\}_{j=1}^\infty$, which is of course bounded. Hence there exists a subsequence $\{k_{j_i}\}_{i=1}^\infty$ of $\{k_j\}_{j=1}^\infty$ such that $p_{12}^{(k_{j_i})} \rightarrow p_{12} \in \mathbb{R}$ ($i \rightarrow +\infty$). Here we still have $p_{11}^{(k_{j_i})} \rightarrow p_{11}$. So we might write k_j instead of k_{j_i} for simplicity. By induction we can obtain $p_{\alpha\beta}$ for all α and β ($\alpha = 1, \dots, l$; $\beta = 1, \dots, m$) so that $p_{\alpha\beta}^{(k_j)} \rightarrow p_{\alpha\beta}$ ($j \rightarrow +\infty$), where $\{k_j\}_{j=1}^\infty$ is some subsequence of the natural number sequence. Let $P = \left(p_{\alpha\beta} \right)_{l \times m}$. Then $P_{k_j} \rightarrow P$ ($j \rightarrow +\infty$).

2.2.6. Definition. Let B be a nonempty subset of \mathbb{R}^n . If $\{A_k\}_{k=1}^\infty$ is a sequence of nonempty subsets of \mathbb{R}^n that satisfies $A_k = \{a_k(b) : b \in B\}$ ($k = 1, 2, \dots$), then $\{A_k\}_{k=1}^\infty$ is said to be a B -index sequence. If further there exists a positive number K so that when $k > K$ we have $d(a_k(b), f(b)) < \varepsilon$ for all $b \in B$, then we say that $\{a_k(b)\}_{k=1}^\infty$ is *uniformly convergent* to $f(b)$ on B , which is denoted by $d(a_k(b), f(b)) \xrightarrow{B} 0$ ($k \rightarrow +\infty$) or $a_k(b) \xrightarrow{B} f(b)$ ($k \rightarrow +\infty$).

2.2.7. Lemma. Let $\{A_k\}_{k=1}^\infty$ be a B -index sequence, where B is a nonempty subset of \mathbb{R}^n and $A_k = \{a_k(b) : b \in B\}$. If $a_k(b) \xrightarrow{B} f(b)$ ($k \rightarrow +\infty$), then

$$h(A_k, f(B)) \rightarrow 0 \quad (k \rightarrow +\infty).$$

Proof. Given any $\varepsilon > 0$, there exists $K > 0$ such that when $k > K$ we have

$$d(a_k(b), f(b)) < \varepsilon$$

for all $b \in B$. Hence for any $b \in B$ we deduce

$$d(a_k(b), f(B)) = \inf_{x \in B} \{d(a_k(b), f(x))\} \leq d(a_k(b), f(b)) < \varepsilon$$

and

$$d(A_k, f(b)) = \inf_{x \in B} \{d(a_k(x), f(b))\} \leq d(a_k(b), f(b)) < \varepsilon.$$

These imply

$$\sup_{b \in B} d(a_k(b), f(B)) \leq \varepsilon$$

and

$$\sup_{b \in B} d(A_k, f(b)) \leq \varepsilon.$$

Therefore

$$h(A_k, f(B)) = \max \left\{ \sup_{b \in B} d(a_k(b), f(B)), \sup_{b \in B} d(A_k, f(b)) \right\} \leq \varepsilon.$$

By Linear Algebra it follows that

2.2.8. Lemma. The transformation $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is an isometry if and only if there are an orthogonal transformation $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a translation $\tau : \mathbb{R}^n \rightarrow \mathbb{R}^n$ satisfying $\varphi = \sigma \circ \tau$ ($\sigma \circ \tau$ denotes the composite of σ and τ).

2.2.9. Lemma. *If $O \in \mathbb{R}^{n \times n}$ is an orthogonal matrix then*

$$|O| \leq n^2.$$

Proof. Let $O = (a_{ij})_{n \times n}$. Then by $OO' = I_n$ (the identity matrix), where O' denotes the transpose of O , we get

$$\sum_{j=1}^n a_{ij}^2 = 1 \quad (i = 1, \dots, n).$$

Hence $a_{ij}^2 \leq 1$, i.e., $|a_{ij}| \leq 1$. Therefore

$$\|O\| = \sum_{i=1}^n \sum_{j=1}^n |a_{ij}| \leq n^2.$$

(Continuation of the proof of Theorem 2.2.) Suppose that $A, B \in \mathcal{C}(\mathbb{R}^n)$ and $\tilde{h}(\tilde{A}, \tilde{B}) = 0$. Then for $k \in \mathbb{P}$ there are $B_k \sim B$ such that

$$H(A, B_k) < \frac{1}{k}.$$

So there exist isometries $\varphi_k = \sigma_k \circ \tau_k$ with $B_k = \varphi_k(B)$, where σ_k are orthogonal transformations and τ_k are translations. Hence

$$B_k = \{\varphi_k(b) : b \in B\},$$

and $\{B_k\}_{k=1}^\infty$ is a B -index sequence. Since

$$\varphi_k(b) = \sigma_k(\tau_k(b)) = (b + t_k)O_k,$$

where $t_k \in \mathbb{R}^n$ and O_k is an orthogonal matrix, we get

$$\begin{aligned} |b + t_k| &= |\sigma_k(b + t_k)| = |\varphi_k(b)| \leq |\varphi_k(b) - a| + |a| \\ &\leq 1 + d(\varphi_k(b), A) + |a| \leq 1 + h(A, B_k) + |a| < 2 + |a|, \end{aligned}$$

where $|x|$ denotes the length of the vector corresponding to $x \in \mathbb{R}^n$ and $a \in A$ is chosen suitably. Therefore

$$|t_k| \leq |-b| + |b + t_k| \leq |b| + 2 + |a| \leq 2R + 2,$$

where $R = \max\{|a|, |b| : a \in A, b \in B\}$. Thus there is a subsequence $\{t_{k_j}\}_{j=1}^\infty$ of $\{t_k\}_{k=1}^\infty$ with $t_{k_j} \rightarrow t \in \mathbb{R}^n$ ($j \rightarrow +\infty$). By Lemma 2.2.9 we have $\|O_{k_j}\| \leq n^2$. Again by Lemma 2.2.5 we obtain a subsequence of $\{O_{k_j}\}_{j=1}^\infty$, still denoted

by $\{\mathbf{O}_{k_j}\}_{j=1}^\infty$, which approaches $\mathbf{O} \in \mathbb{R}^{n \times n}$. Hence $\mathbf{O}'_{k_j} \rightarrow \mathbf{O}'$ ($j \rightarrow +\infty$). By Lemma 2.2.4 it follows

$$\mathbf{O}_{k_j} \mathbf{O}'_{k_j} \rightarrow \mathbf{O} \mathbf{O}'.$$

As $\mathbf{O}_{k_j} \mathbf{O}'_{k_j} = \mathbf{I}_n$, we have $\mathbf{O} \mathbf{O}' = \mathbf{I}_n$, which means \mathbf{O} is orthogonal. Let $\varphi = \sigma \circ \tau$, where $\sigma(x) = x\mathbf{O}$ and $\tau(x) = x + t$. Then φ is an isometry.

Since

$$\begin{aligned} d(\varphi_{k_j}(b), \varphi(b)) &= |\varphi_{k_j}(b) - \varphi(b)| \\ &\leq \|\varphi_{k_j}(b) - \varphi(b)\| = \|\tau_{k_j}(b)\mathbf{O}_{k_j} - \tau(b)\mathbf{O}\| \\ &\leq \|\tau_{k_j}(b) - \tau(b)\| \|\mathbf{O}_{k_j}\| + \|\tau(b)\| \|\mathbf{O}_{k_j} - \mathbf{O}\| \\ &\leq n^2 \|t_{k_j} - t\| + M \|\mathbf{O}_{k_j} - \mathbf{O}\| \rightarrow 0 \quad (j \rightarrow +\infty), \end{aligned}$$

where M relies on A and B but does not rely on $b \in B$, we have

$$\varphi_{k_j}(b) \xrightarrow{B} \varphi(b) \quad (j \rightarrow +\infty).$$

By Lemma 2.2.7 it follows

$$h(\varphi_{k_j}(B), \varphi(b)) \rightarrow 0 \quad (j \rightarrow +\infty).$$

Therefore

$$h(A, \varphi(B)) \leq h(A, B_{k_j}) + h(\varphi_{k_j}(B), \varphi(b)) \rightarrow 0 \quad (j \rightarrow +\infty).$$

So $h(A, \varphi(B)) = 0$. This implies $A = \varphi(B)$, i.e., $\tilde{A} = \tilde{B}$.

Now let us prove that the metric \tilde{h} is complete. Suppose $\{\tilde{A}_k\}_{k=1}^\infty$ is a Cauchy sequence in $\tilde{\mathcal{C}}(\mathbb{R}^n)$. Then for an arbitrary $\varepsilon > 0$, there exists $K > 0$ such that when $j, k \geq K$ one has $\tilde{h}(\tilde{A}_j, \tilde{A}_k) < \varepsilon$.

(a) Claim. *One may select a subsequence $\{\tilde{A}_{k_i}\}_{i=1}^\infty$ of $\{\tilde{A}_k\}_{k=1}^\infty$ such that*

$$\tilde{h}(\tilde{A}_{k_i}, \tilde{A}_{k_{i+1}}) < 2^{-i}$$

for $i = 1, 2, \dots$.

Proof. We can easily know that there exists a subsequence $\{\tilde{A}_{k_i}\}_{i=1}^\infty$ ($k_1 < k_2 < \dots < k_i < \dots$) so that

$$\tilde{h}(\tilde{A}_{k_i}, \tilde{A}_{k_i}) < 2^{-i}$$

hold for all $k > k_i$, where $i = 1, 2, \dots$. Taking $k = k_{i+1}$ the claim follows.

(b) Claim. *There exist $B_j \in \mathcal{C}(\mathbb{R}^n)$ so that $B_j \sim A_{k_j}$ and*

$$h(B_j, B_{j+1}) < 2^{-j} \quad (j = 1, 2, \dots).$$

Proof. Let $B_1 = A_{k_1}$. From $\tilde{h}(\tilde{A}_{k_1}, \tilde{A}_{k_2}) < 2^{-1}$ we get that there exists $B_2 \sim A_{k_2}$ such that

$$h(A_{k_1}, B_2) < 2^{-1}.$$

Suppose there exist $B_i \sim A_{k_i}$ ($i = 1, 2, \dots, j$) such that

$$h(B_i, B_{i+1}) < 2^{-i} \quad (i = 1, 2, \dots, j-1).$$

Then by Claim (a) we have

$$\tilde{h}(\tilde{B}_j, \tilde{A}_{k_{j+1}}) = \tilde{h}(\tilde{A}_{k_j}, \tilde{A}_{k_{j+1}}) < 2^{-j}.$$

So we may choose $B_{j+1} \sim A_{k_{j+1}}$ such that

$$h(B_j, B_{j+1}) < 2^{-j}.$$

By induction Claim (b) holds.

(c) Claim. *Let $C_k = \text{cl} \left(\bigcup_{j \geq k} B_j \right)$ ($\text{cl}(A)$ indicates the closure of A) and $C = \bigcap_{k=1}^{\infty} C_k$. Then*

$$h(C_k, B_k) \rightarrow 0 \quad \text{and} \quad h(C_k, C) \rightarrow 0 \quad (k \rightarrow +\infty).$$

Proof. Let $k \in \mathbb{P}$. It is obvious that $C_k \supseteq C_{k+1}$ and C is a nonempty closed set. By Claim (b) it follows that

$$h(B_j, B_{j+p}) \leq \sum_{i=j}^{j+p-1} h(B_i, B_{i+1}) < \sum_{i=j}^{j+p-1} 2^{-i} < 2^{-j+1}$$

for all $p \in \mathbb{P}$. Hence $\{B_j\}_{j=1}^{\infty}$ is a Cauchy sequence. As $h(B_1, B_j) < 1$ we know that $B_j \subseteq \mathcal{P}(B_1, 1)$ ($j \in \mathbb{P}$). So

$$C_k = \text{cl} \left(\bigcup_{j \geq k} B_j \right) \subseteq \mathcal{P}(B_1, 1),$$

which implies $C_k \in \mathcal{C}(\mathbb{R}^n)$ ($k \in \mathbb{P}$) and $C \in \mathcal{C}(\mathbb{R}^n)$. Similarly we have $B_j \subseteq \mathcal{P}(B_k, 2^{-k+1})$ ($j \geq k$). Thus

$$C_k = \text{cl} \left(\bigcup_{j \geq k} B_j \right) \subseteq \mathcal{P}(B_k, 2^{-k+1}).$$

This implies

$$h(C_k, B_k) \leq 2^{-k+1} \rightarrow 0.$$

Therefore

$$\begin{aligned} h(C_k, C_{k+p}) &\leq h(C_k, B_k) + h(B_k, B_{k+p}) + h(B_{k+p}, C_{k+p}) \\ &< 2^{-k+1} + 2^{-k+1} + 2^{-k-p+1} < 2^{-k+3}. \end{aligned}$$

So we have

$$C_k \subseteq \mathcal{P}(C_{k+p}, 2^{-k+3}) \quad (p \in \mathbb{P}).$$

Suppose $x \in C_k$. Then there is $x_p \in C_{k+p}$ so that $d(x, x_p) \leq 2^{-k+3}$. We choose a subsequence $\{x_{p_i}\}_{i=1}^\infty$ of $\{x_p\}_{p=1}^\infty$ such that $x_{p_i} \rightarrow x_0 \in \mathbb{R}^n$ ($i \rightarrow +\infty$). Then $x_0 \in C_{k+p_i}$. Hence $x_0 \in C$ and

$$d(x, x_0) \leq 2^{-k+3},$$

which means $x \in \mathcal{P}(C, 2^{-k+3})$. Consequently $C_k \subseteq \mathcal{P}(C, 2^{-k+3})$. Therefore

$$h(C_k, C) \leq 2^{-k+1} \rightarrow 0,$$

and Claim (c) follows.

Given $\varepsilon > 0$, by Claim (c) there exists $K > 0$ so that we may take a sufficiently great j , when $k > K$ we have

$$\begin{aligned} \tilde{h}(\tilde{A}_k, \tilde{C}) &\leq \tilde{h}(\tilde{A}_k, \tilde{A}_{k_j}) + \tilde{h}(\tilde{B}_j, \tilde{C}_j) + \tilde{h}(\tilde{C}_j, \tilde{C}) \\ &\leq \tilde{h}(\tilde{A}_k, \tilde{A}_{k_j}) + h(\tilde{B}_j, \tilde{C}_j) + h(\tilde{C}_j, \tilde{C}) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \end{aligned}$$

Finally $\tilde{h}(\tilde{A}_k, \tilde{C}) \rightarrow 0$ ($k \rightarrow +\infty$). This completes the proof of Theorem 2.2. \square

2.3. Remark. (1) The metric space $(\tilde{\mathcal{C}}(\mathbb{R}^n), \tilde{h})$ is separable.

(2) We may regard $(\mathcal{C}(\mathbb{R}^n), \tilde{h})$ as $(\tilde{\mathcal{C}}(\mathbb{R}^n), \tilde{h})$ and “ \sim ” as “ $=$ ” in the definition of metric spaces. Under this convention, we may say that $(\mathcal{C}(\mathbb{R}^n), \tilde{h})$ is a complete metric space.

(3) By a similar reasoning to that in the proof of Theorem 2.2 we may show

$$\tilde{h}(A, B) = \min \{h(A_1, B_1) : A_1 \sim A, B_1 \sim B\}.$$

Starting from this conclusion we can also deduce Theorem 2.2.

2.4. Rigid shape differences.

2.4.1. Definition. An isometry ψ in \mathbb{R}^n is said to be *rigid* if $\psi = \sigma \circ \tau$, where τ is a translation and σ is a rigid orthogonal transformation or rotation, i.e., $\det \sigma = 1$ ($\det \sigma := \det \mathbf{O}$ where \mathbf{O} is a matrix of σ under some orthonormal basis of \mathbb{R}^n). We also call a rigid isometry a *rigid motion*.

2.4.2. Definition. Two nonempty subsets A and B of \mathbb{R}^n are *rigidly equivalent* if there exists a rigid isometry ψ in \mathbb{R}^n such that $B = \psi(A)$, denoted by $A \simeq B$.

2.4.3. Lemma. Let $a \in \mathbb{R}^n$ and define $\tau_a := x + a$ for $x \in \mathbb{R}^n$. Suppose σ is an orthogonal transformation, ψ_1 and ψ_2 are rigid isometries and $b \in \mathbb{R}^n$. Then

- (1) $\tau_a \circ \tau_b = \tau_{a+b}$;
- (2) $\tau_a^{-1} = \tau_{-a}$;
- (3) $\tau_a \circ \sigma = \sigma \circ \tau_{\sigma^{-1}(a)}$;
- (4) ψ_1^{-1} and $\psi_2 \circ \psi_1$ are also rigid isometries.

Proof. (1) and (2) are obvious. For (3) and (4), letting $x \in \mathbb{R}^n$ we have

$$\tau_a \circ \sigma(x) = \sigma(x) + a = \sigma(x + \sigma^{-1}(a)) = \sigma \circ \tau_{\sigma^{-1}(a)}(x).$$

Let $\psi_1 = \sigma_1 \circ \tau_a$ and $\psi_2 = \sigma_2 \circ \tau_b$, where σ_1 and σ_2 are rigid orthogonal transformations. Then σ_1^{-1} and $\sigma_2 \circ \sigma_1$ are rigid orthogonal transformations,

$$\psi_1^{-1} = \tau_a^{-1} \circ \sigma_1^{-1} = \tau_{-a} \circ \sigma_1^{-1} = \sigma_1^{-1} \circ \tau_{-\sigma_1(a)}$$

and

$$\begin{aligned} \psi_2 \circ \psi_1 &= (\sigma_2 \circ \tau_b) \circ (\sigma_1 \circ \tau_a) = \sigma_2 \circ (\tau_b \circ \sigma_1) \circ \tau_a \\ &= \sigma_2 \circ (\sigma_1 \circ \tau_{\sigma_1^{-1}(b)}) \circ \tau_a = (\sigma_2 \circ \sigma_1) \circ \tau_{a + \sigma_1^{-1}(b)}. \end{aligned} \quad \square$$

2.4.4. By Lemma 2.4.3(4) we see that the relation “ \simeq ” is an equivalent relation. The equivalence class of A is denoted \overline{A} . And we denote

$$\overline{\mathcal{C}}(\mathbb{R}^n) = \{\overline{C} : C \in \mathcal{C}(\mathbb{R}^n)\}.$$

2.4.5. Definition. For nonempty subsets A and B of \mathbb{R}^n we define

$$\overline{h}(A, B) := \inf \{h(\varphi(A), \psi(B)) : \varphi \text{ and } \psi \text{ are rigid isometries in } \mathbb{R}^n\}.$$

By Lemmas 2.1.2 and 2.4.3(4) we have the following expressions:

$$\begin{aligned} \overline{h}(A, B) &:= \inf \{h(A_1, B_1) : A_1 \simeq A, B_1 \simeq B\} \\ &= \inf \{h(A_0, B_1) : B_1 \simeq B\} \quad (\text{where } A_0 \simeq A) \\ &= \inf \{h(A_1, B_0) : A_1 \simeq A\} \quad (\text{where } B_0 \simeq B). \end{aligned}$$

If $A_1 \simeq A_2$ and $B_1 \simeq B_2$, then $\overline{h}(A_1, B_1) = \overline{h}(A_2, B_2)$. So we may define

$$\overline{h}(\overline{A}, \overline{B}) := \overline{h}(A, B),$$

which is called the (*absolute*) *rigid shape difference* between \tilde{A} and \tilde{B} or between A and B .

2.5. Theorem. $(\overline{\mathcal{C}}(\mathbb{R}^n), \overline{h})$ is a complete metric space.

The proof of Theorem 2.5 is just similar to that of Theorem 2.2, where we use Definition 2.4.1 instead of Lemma 2.2.8. \square

2.6. Relative shape differences.

2.6.1. Definition. Let $r > 0$.

(1) A transformation $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a *similitude* or *r-similitude* if

$$d(S(x), S(y)) = rd(x, y)$$

for all $x, y \in \mathbb{R}^n$, and r is called the *Lipschitz constant* of S .

(2) Two nonempty subsets A and B of \mathbb{R}^n are *similar* if there exists a similitude $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that $B = S(A)$, denoted by $A \widehat{\sim} B$. We easily know that this relation is an equivalence relation. The equivalence class of A is denoted \widehat{A} .

(3) The transformation $\mu_r(x) = rx$ ($x \in \mathbb{R}^n$) is called a *homothety* or *r-homothety*.

2.6.2. Definition. Let A be a nonempty subset of \mathbb{R}^n . The *diameter* of A is

$$|A| := \sup\{d(x, y) : x, y \in A\}.$$

We define the *radius* $r(A)$ of A by

$$r(A) := \inf_{x \in \mathbb{R}^n} \left\{ \sup_{a \in A} d(x, a) \right\}.$$

We also denote $r(A)$ by r_A . Denote

$$\rho A := \{\rho a : a \in A\}$$

for $\rho \geq 0$. Generally $r(A) \neq \frac{1}{2}|A|$.

For the need later on, we list the following conclusions.

2.6.3. Proposition. Let A and B be nonempty bounded subsets of \mathbb{R}^n .

- (1) $r(S(A)) = r r(A)$ for a similitude S of Lipschitz constant r ; specially $r(\sigma(A)) = r(A)$ for an isometry σ .
- (2) $||A| - |B|| \leq 2\widetilde{h}(A, B)$.
- (3) $|r(A) - r(B)| \leq h(A, B)$.

$$(4) \quad |r(A) - r(B)| \leq \tilde{h}(A, B).$$

(5) *There exists $x_0 \in \mathbb{R}^n$ such that*

$$\sup_{a \in A} d(x_0, a) = r(A).$$

We call x_0 a *center* of A (it is possible that $x_0 \notin A$).

(6) *If $0 < \rho < +\infty$ then $h(A, \rho A) \leq |\rho - 1|r(A)$.*

Proof. (1) is obvious and (4) follows from (1) and (3). The proof of (2) is similar to that of (4) and easier. Now we prove (3), (5) and (6).

(i) Let ε be an arbitrary positive number. Given any $x \in \mathbb{R}^n$ we have

$$r(A) \leq \sup_{a \in A} d(x, a).$$

Thus there exists $a \in A$ such that

$$r(A) - \varepsilon < d(x, a).$$

Choose $b_0 \in B$ so that

$$d(a, b_0) < d(a, B) + \varepsilon.$$

Then

$$\begin{aligned} r(A) - \varepsilon &< d(x, b_0) + d(b_0, a) \\ &< \sup_{b \in B} d(x, b) + d(a, B) + \varepsilon \leq \sup_{b \in B} d(x, b) + h(A, B) + \varepsilon. \end{aligned}$$

It follows that

$$r(A) - 2\varepsilon \leq \inf_{x \in \mathbb{R}^n} \left\{ \sup_{b \in B} d(x, b) \right\} + h(A, B) = r(B) + h(A, B).$$

Therefore

$$r(A) \leq r(B) + h(A, B).$$

Hence (3) is true.

(ii) For any positive integers k there exist $x_k \in \mathbb{R}^n$ such that

$$\sup_{a \in A} d(x_k, a) < r(A) + \frac{1}{k}.$$

Thus there exist $x_0 \in \mathbb{R}^n$ and a subsequence $\{x_{k_j}\}_{j=1}^\infty$ of $\{x_k\}_{k=1}^\infty$ such that

$$d(x_{k_j}, x_0) \rightarrow 0 \quad (j \rightarrow +\infty).$$

Therefore

$$\sup_{a \in A} d(x_0, a) \leq r(A).$$

(iii) Suppose x_0 is a center of A . Let

$$B = A - x_0 := \{a - x_0 : a \in A\}.$$

Then

$$\rho B = \rho A - \rho x_0 := \{\rho a - \rho x_0 : a \in A\}.$$

Since

$$d(b, \rho B), d(\rho b, B) \leq |b - \rho b| = |\rho - 1| |b| \leq |\rho - 1| r(A),$$

we have

$$\begin{aligned} \tilde{h}(A, \rho A) &= \tilde{h}(B, \rho B) \\ &\leq \sup\{d(b, \rho B), d(\rho b, B) : b \in B\} \leq |\rho - 1| r(A). \end{aligned} \quad \square$$

2.6.4. Lemma. *Let A and B be nonempty bounded subsets of \mathbb{R}^n . Then $\hat{A} = \hat{B}$ if and only if $\frac{A}{r(A)} \sim \frac{B}{r(B)}$, where if A is a singleton then we treat $\frac{A}{r(A)}$ as A .*

Proof. We easily know that an isometry is a 1-similitude, an r -homothety is an r -similitude ($r > 0$), and if S_1 and S_2 are an r_1 -similitude and an r_2 -similitude respectively then $S_1 \circ S_2$ is an $r_1 r_2$ -similitude. When A or B is a singleton, the lemma is obviously true. Now suppose neither A nor B is a singleton.

If $\frac{A}{r(A)} \sim \frac{B}{r(B)}$, then $\frac{B}{r(B)} = \varphi \left(\frac{A}{r(A)} \right)$, where φ is an isometry. Hence

$$B = r(B) \varphi \left(\frac{1}{r(A)} A \right) = \left(\mu_{r(B)} \circ \varphi \circ \mu_{(r(A))^{-1}} \right) (A),$$

which means $A \hat{\sim} B$.

If $\hat{A} = \hat{B}$, i.e., $A \hat{\sim} B$, then $B = S(A)$, where S is an r -similitude ($r > 0$), so $r(B) = r r(A)$. Consequently

$$\frac{B}{r(B)} = \frac{1}{r r(A)} S(A) = \left(\mu_{(r r(A))^{-1}} \circ S \circ \mu_{r(A)} \right) \left(\frac{A}{r(A)} \right),$$

where $\mu_{(r r(A))^{-1}} \circ S \circ \mu_{r(A)}$ is an isometry. \square

2.6.5. Definition. Let A and B be nonempty bounded subsets of \mathbb{R}^n . Define

$$\widehat{h}(A, B) := \widetilde{h}\left(\frac{A}{r(A)}, \frac{B}{r(B)}\right)$$

and

$$\widehat{h}(\widehat{A}, \widehat{B}) := \widehat{h}(A, B),$$

which is called the *relative shape difference* between \widehat{A} and \widehat{B} or between A and B .

Remark. By Lemma 2.6.4 we see that if $A \widehat{\sim} A_1$ and $B \widehat{\sim} B_1$ then

$$\frac{A}{r(A)} \sim \frac{A_1}{r(A_1)} \quad \text{and} \quad \frac{B}{r(B)} \sim \frac{B_1}{r(B_1)},$$

so

$$\widehat{h}(A, B) = \widetilde{h}\left(\frac{A}{r(A)}, \frac{B}{r(B)}\right) = \widetilde{h}\left(\frac{A_1}{r(A_1)}, \frac{B_1}{r(B_1)}\right) = \widehat{h}(A_1, B_1).$$

This implies that the above definition of $\widehat{h}(\widehat{A}, \widehat{B})$ is well-defined.

2.6.6. Denote

$$\widehat{\mathcal{C}}(\mathbb{R}^n) = \left\{ \widehat{C} : C \in \mathcal{C}(\mathbb{R}^n) \right\}.$$

2.7. Theorem. $(\widehat{\mathcal{C}}(\mathbb{R}^n), \widehat{h})$ is a complete metric space.

Proof. Let $A, B, C \in \mathcal{C}(\mathbb{R}^n)$. It is obvious that

$$\widehat{h}(\widehat{A}, \widehat{B}) = \widehat{h}(\widehat{B}, \widehat{A}) \geq 0.$$

By Lemma 2.6.4 we see that $\widehat{A} = \widehat{B}$ if and only if

$$\frac{A}{r(A)} \sim \frac{B}{r(B)},$$

which is equivalent to

$$\widetilde{h}\left(\frac{A}{r(A)}, \frac{B}{r(B)}\right) = 0,$$

i.e., $\widehat{h}(\widehat{A}, \widehat{B}) = 0$.

According to Definition 2.6.5 and Theorem 2.2 we have

$$\begin{aligned} \widehat{h}(\widehat{A}, \widehat{B}) &= \widetilde{h}\left(\frac{A}{r(A)}, \frac{B}{r(B)}\right) \\ &\leq \widetilde{h}\left(\frac{A}{r(A)}, \frac{C}{r(C)}\right) + \widetilde{h}\left(\frac{C}{r(C)}, \frac{B}{r(B)}\right) = \widehat{h}(\widehat{A}, \widehat{C}) + \widehat{h}(\widehat{C}, \widehat{B}). \end{aligned}$$

Now suppose $\{\widehat{A}_k\}_{k=1}^\infty$ is a Cauchy sequence in $(\widehat{\mathcal{C}}(\mathbb{R}^n), \widehat{h})$. Then $\{\widetilde{B}_k\}_{k=1}^\infty$, where $B_k = \frac{A_k}{r(A_k)}$, is a Cauchy sequence in $(\widetilde{\mathcal{C}}(\mathbb{R}^n), \widetilde{h})$ by Definition 2.6.5, which implies by Theorem 2.2 that there exists $A \in \mathcal{C}(\mathbb{R}^n)$ such that

$$\widetilde{h}(\widetilde{B}_k, \widetilde{A}) \rightarrow 0 \quad (k \rightarrow +\infty).$$

For any $\varepsilon > 0$, there exists $K > 0$ such that when $k > K$ we have

$$\widetilde{h}(\widetilde{B}_k, \widetilde{A}) < \varepsilon.$$

By Proposition 2.6.3(4) it follows that

$$1 - \varepsilon < r(A) < 1 + \varepsilon.$$

Thus $r(A) = 1$. Consequently

$$\widehat{h}(\widehat{A}_k, \widehat{A}) = \widetilde{h}(\widetilde{B}_k, \widetilde{A}) \rightarrow 0 \quad (k \rightarrow +\infty). \quad \square$$

2.8. Relative rigid shape differences. By a normal reasoning we know that an r -similitude $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ can just be expressed to be $S = \mu_r \circ \sigma \circ \tau_a$, where μ_r is an r -homothety, σ is an orthogonal transformation and τ_a is a translation (see [53, Proposition 2.3(1)]). Now let us give the following

2.8.1. Definition. A similitude $S : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is called a *rigid r -similitude* ($r > 0$) if $S = \mu_r \circ \sigma \circ \tau_a$, where σ is a rotation, i.e., $\det \sigma = 1$ (cf. Definition 2.4.1).

2.8.2. Lemma. Let r, r_1, r_2 be positive real numbers.

(1) Let μ_r be an r -homothety and let $\sigma, \sigma_1, \sigma_2$ be orthogonal transformations. Let τ_a ($a \in \mathbb{R}^n$) be a translation defined in Lemma 2.4.3. Then

- (i) $\mu_r \circ \sigma = \sigma \circ \mu_r$;
- (ii) $\mu_r \circ \tau_a = \tau_{ra} \circ \mu_r$;
- (iii) $\mu_{r_1} \circ \mu_{r_2} = \mu_{r_1 r_2}$;
- (iv) $\mu_r^{-1} = \mu_{r^{-1}}$;
- (v) $\sigma_1 \circ \sigma_2$ is an orthogonal transformation, and $\det \sigma_1 = \det \sigma_2 = 1$ implies $\det(\sigma_1 \circ \sigma_2) = 1$;
- (vi) σ^{-1} is an orthogonal transformation, and $\det \sigma = 1$ if and only if $\det \sigma^{-1} = 1$.

(2) Let S_i be a rigid r_i -similitude in \mathbb{R}^n ($i = 1, 2$). Then

- (i) *identical mapping* $\text{id} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a rigid 1-similitude;
- (ii) S_1^{-1} is a rigid r_1^{-1} -similitude;
- (iii) $S_1 \circ S_2$ is a rigid $r_1 r_2$ -similitude.

Proof. (1) and (2)(i) are clear. Suppose $S_1 = \mu_{r_1} \circ \sigma_1 \circ \tau_{a_1}$ and $S_2 = \mu_{r_2} \circ \sigma_2 \circ \tau_{a_2}$. Then by (1) and Lemma 2.4.3,

$$\begin{aligned} S_1^{-1} &= \tau_{a_1}^{-1} \circ \sigma_1^{-1} \circ \mu_{r_1}^{-1} = \tau_{-a_1} \circ \sigma_1^{-1} \circ \mu_{r_1^{-1}} \\ &= \tau_{-a_1} \circ \mu_{r_1^{-1}} \circ \sigma_1^{-1} = \mu_{r_1^{-1}} \circ \tau_{-r_1 a_1} \circ \sigma_1^{-1} \\ &= \mu_{r_1^{-1}} \circ \sigma_1^{-1} \circ \tau_{-r_1 \sigma_1(a_1)} \end{aligned}$$

is a rigid r_1^{-1} -similitude; and

$$\begin{aligned} S_1 \circ S_2 &= \mu_{r_1} \circ \sigma_1 \circ \tau_{a_1} \circ \mu_{r_2} \circ \sigma_2 \circ \tau_{a_2} \\ &= \mu_{r_1} \circ \sigma_1 \circ \mu_{r_2} \circ \sigma_2 \circ \tau_{r_2^{-1} \sigma_2^{-1}(a_1)} \circ \tau_{a_2} \\ &= \mu_{r_1 r_2} \circ (\sigma_1 \circ \sigma_2) \circ \tau_{a_2 + r_2^{-1} \sigma_2^{-1}(a_1)} \end{aligned}$$

is a rigid $r_1 r_2$ -similitude. □

2.8.3. Definition. Two nonempty subsets A and B of \mathbb{R}^n are *rigidly similar* if there exists a rigid similitude S in \mathbb{R}^n such that $B = S(A)$, denoted by $A \sim B$. By Lemma 2.8.2(2) we know that the rigid similarity is an equivalence relation. The equivalence class of A is denoted \check{A} . And we denote

$$\check{\mathcal{C}}(\mathbb{R}^n) = \{ \check{C} : C \in \mathcal{C}(\mathbb{R}^n) \}.$$

By Lemma 2.8.2 and similarly to Lemma 2.6.4 we may get

2.8.4. Lemma. Let A and B be nonempty bounded subsets of \mathbb{R}^n . Then $\frac{A}{r(A)} \simeq \frac{B}{r(B)}$ if and only if $\check{A} = \check{B}$. □

2.8.5. Definition. Let A and B be nonempty bounded subsets of \mathbb{R}^n . Define

$$\check{h}(A, B) := \bar{h} \left(\frac{A}{r(A)}, \frac{B}{r(B)} \right)$$

and

$$\check{h}(\check{A}, \check{B}) := \check{h}(A, B),$$

which is called the *relative rigid shape difference* between \check{A} and \check{B} or between A and B .

Remark. By Lemma 2.8.4 the above definition of $\check{h}(\check{A}, \check{B})$ is well-defined.

2.9. Theorem. $(\check{\mathcal{C}}(\mathbb{R}^n), \check{h})$ is a complete metric space.

The proof of this theorem is similar to that of Theorem 2.7. \square

2.10. Remark. (1) The metric spaces $(\overline{\mathcal{C}}(\mathbb{R}^n), \overline{h})$, $(\widehat{\mathcal{C}}(\mathbb{R}^n), \widehat{h})$ and $(\check{\mathcal{C}}(\mathbb{R}^n), \check{h})$ are all separable.

(2) We may regard $(\mathcal{C}(\mathbb{R}^n), \overline{h})$, $(\mathcal{C}(\mathbb{R}^n), \widehat{h})$ and $(\mathcal{C}(\mathbb{R}^n), \check{h})$ as $(\overline{\mathcal{C}}(\mathbb{R}^n), \overline{h})$, $(\widehat{\mathcal{C}}(\mathbb{R}^n), \widehat{h})$ and $(\check{\mathcal{C}}(\mathbb{R}^n), \check{h})$ respectively; and regard “ \simeq ”, “ $\widehat{\sim}$ ” and “ $\check{\sim}$ ” as “ $=$ ” correspondingly in the definition of metric spaces. Under this convention, we may say that $(\mathcal{C}(\mathbb{R}^n), \overline{h})$, $(\mathcal{C}(\mathbb{R}^n), \widehat{h})$ and $(\mathcal{C}(\mathbb{R}^n), \check{h})$ are complete metric spaces.

(3) In Definition 2.6.5 and 2.8.5 we may define the relative shape difference and relative rigid shape difference using diameters instead of radii, i.e., define

$$\widehat{h}(A, B) := \widetilde{h}\left(\frac{A}{|A|}, \frac{B}{|B|}\right) \quad \text{and} \quad \check{h}(A, B) := \overline{h}\left(\frac{A}{|A|}, \frac{B}{|B|}\right),$$

and obtain similar results.

(4) We may also define the translation shape difference by using translation equivalence instead of isometric equivalence, and obtain some similar results.

3 Perturbation of self-similar sets

In 1946, P. A. P. Moran considered a self-similar set as an extension of Cantor’s set, obtained its Hausdorff dimension and proved it has a finite and positive Hausdorff measure at its Hausdorff dimension when it satisfies the open set condition (see [87]).

Researches into self-similar sets were once motivated by Mandelbrot’s work (see [72] and [73]). In 1981, J. Hutchinson ([53]) considered a self-similar set as an invariant set of a finite set of contraction maps (similitudes) (called an *iterated function system*) in a systematic manner and a mathematical self-similar set is presented in a clear and wonderful way. In [53] he also considered an invariant measure with respect to the iterated function system. Later on a large number of researches have been done in the area of self-similarity and related subjects.

Researches on self-similarities have been developed in many directions. Separation properties for self-similar sets have been considered in [96] and [111], etc. For researches of iterated function systems and some related topics, we refer to [3], [5], [6], [7], [9], [35], [53] and [54], etc. If the iterated function system consists of affine transformations then the invariant set is a self-affine set, which is an extension of a self-similar set and has been investigated extensively (see

e.g. [5], [11], [13], [30], [32] and [82], etc). Infinite iterated function systems have also been considered (see e.g. [42], [78] and [86], etc). For researches on random cases we refer to e.g. [8], [9], [29], [45], [46], [54], [55] and [79], etc. For Moran sets, which are extensions of self-similar sets, we refer to e.g. [21], [41], [52], [68], [69], [109, Chapter 8] and [110], etc. For graph directed constructions we refer to e.g. [25] and [80], etc. For sub-self-similar sets we refer to e.g. [33] and [34, Section 3.1], etc. We note that Dekking ([26]) once gave a recurrent structure method to construct some fractals. A kind of quasi-self-similar sets has been considered in e.g. [15, Theorem 8.6], [31], [81] and [104, p.742], etc.

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In this section we consider a kind of fractals which can be approximately regarded as self-similar sets and we deal with them as perturbation of strict self-similar sets.

3.1. Sequences of integers. We call $\mathbf{i} = \mathbf{i}_k = i_1 \cdots i_k$, where $i_j \in \mathbb{P}$ (the set of positive integers) ($j = 1, \dots, k$), a *word of finite length* (the *length* $|\mathbf{i}| = k$); and call $\alpha = i_1 \cdots i_j \cdots$, where $i_j \in \mathbb{P}$ ($j = 1, 2, \dots$), a *word of infinite length*.

Let $\{m_j\}_{j=1}^\infty$ be a sequence of positive integers and usually $m_j \geq 2$ ($j = 1, 2, \dots$). Let $\mathbf{i} = i_1 \cdots i_k$. We write $\mathbf{j} = \mathbf{i} i_{k+1} \cdots i_l$ if $\mathbf{j} = i_1 \cdots i_k i_{k+1} \cdots i_l$ ($l \geq k$) and write $\alpha = \mathbf{i} i_{k+1} \cdots i_l \cdots$ if $\alpha = i_1 \cdots i_k i_{k+1} \cdots i_l \cdots$. And then we denote $\mathbf{j}|_k = \mathbf{i}$ and $\alpha|_k = \mathbf{i}$.

Now let us define

$$\hat{\mathbf{i}} := \{\mathbf{i} i_{k+1} \cdots i_l \cdots : i_l = 1, \dots, m_l; \quad l = k+1, k+2, \dots\},$$

where $|\mathbf{i}| = k \geq 1$, and we still denote $\hat{\mathbf{i}}$ by \mathbf{i} . Define

$$\begin{aligned} \mathcal{I}_\infty &= \mathcal{I}_\infty(\{m_j\}) \\ &:= \{\alpha = i_1 \cdots i_k \cdots : i_k = 1, \dots, m_k; \quad k = 1, 2, \dots\}, \end{aligned}$$

which is also denoted $\hat{\mathbf{0}}$ or $\mathbf{0}$. Assume $\mathbf{i}|_0 := \mathbf{0}$. Then $\mathbf{i} \supseteq \mathbf{j}$ if and only if $|\mathbf{j}| = l \geq k = |\mathbf{i}|$ and $\mathbf{j}|_k = \mathbf{i}$ ($k \geq 0$). It is obvious that $\alpha \in \mathbf{i}$ if $\alpha|_k = \mathbf{i}$ ($k \geq 1$) and that all $\alpha \in \mathbf{0}$. Let

$$\begin{aligned} \mathcal{I} &= \mathcal{I}(\{m_j\}) \\ &:= \{\mathbf{i} = i_1 \cdots i_k : i_k = 1, \dots, m_k; \quad k = 1, 2, \dots\} \cup \{\mathbf{0}\}, \\ \mathcal{I}_k &= \mathcal{I}_k(\{m_j\}) := \{\mathbf{i} \in \mathcal{I} : |\mathbf{i}| = k\} \quad (k = 0, 1, 2, \dots) \end{aligned}$$

and

$$\mathcal{I}^l = \mathcal{I}^l(\{m_j\}) := \bigcup_{k=0}^l \mathcal{I}_k(\{m_j\}) \quad (l = 0, 1, 2, \dots).$$

If $m_k = m$ for all $k \in \mathbb{P}$ then we say \mathbf{i} and α to be *normal words* and, $\mathcal{I}(\{m_j\})$, $\mathcal{I}_k(\{m_j\})$, $\mathcal{I}^l(\{m_j\})$ and $\mathcal{I}_\infty(\{m_j\})$ are denoted $\mathcal{I}(m)$, $\mathcal{I}_k(m)$, $\mathcal{I}^l(m)$ and $\mathcal{I}_\infty(m)$ respectively.

If some or all of m_j ($j = 1, 2, \dots$) equal $+\infty$, we may also give similar concepts to the above and we will use the same notations to denote them.

3.2. Perturbation of self-similar sets.

3.2.1. Definition. Let $\mathcal{S} = \{S_i : i = 1, \dots, m\}$ be a family of contraction similitudes, which is called an *iterated function system of similitudes* (abbreviated to *IFSS*). Let E be the compact invariant set determined by \mathcal{S} , i.e., $\mathcal{S}(E) = E$, where $\mathcal{S}(E) := \bigcup_{i=1}^m S_i(E)$ (see [35, Chapter 9] and [53]).

Let $\mathcal{F} = \{F_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ be a family of compact sets in \mathbb{R}^n satisfying

$$F_{\mathbf{i}} = \bigcup_{i=1}^m F_{i\mathbf{i}}$$

for all $\mathbf{i} \in \mathcal{I}$. Let $F = F_{\mathbf{0}}$. Then we say that \mathcal{F} is a *structure system* of F . Given a family $\Delta = \{\delta_{\mathbf{i}} \geq 0 : \mathbf{i} \in \mathcal{I}\}$ of nonnegative real numbers, let $\delta = \sup\{\delta_{\mathbf{i}} : \delta_{\mathbf{i}} \in \Delta\}$. Assume $\text{Lip } S_i = c_i$, i.e.,

$$|S_i(x) - S_i(y)| = c_i |x - y|$$

for $x, y \in \mathbb{R}^n$ ($i = 1, \dots, m$). Denote $c_{\mathbf{i}} := c_{i_1} \cdots c_{i_k}$, where $\mathbf{i} = i_1 \cdots i_k$, and $c_{\mathbf{0}} := 1$.

Suppose

$$\tilde{h}(F_{\mathbf{i}}, E_{\mathbf{i}}) \leq \delta_{\mathbf{i}} c_{\mathbf{i}} r(E), \quad (3.1)$$

where $S_{\mathbf{i}} := S_{i_1} \circ \cdots \circ S_{i_k}$ and $E_{\mathbf{i}} := S_{\mathbf{i}}(E)$ for $\mathbf{i} = i_1 \cdots i_k$, $S_{\mathbf{0}} := \text{id}$ (the identical mapping), $r(E)$ is the radius of E . Then F is called a Δ -*perturbation of the self-similar set* E or a Δ -*quasi-self-similar set* with the IFSS \mathcal{S} . If $\delta < +\infty$, then F is also called a δ -*perturbation of the self-similar set* E or a δ -*quasi-self-similar set* with the IFSS \mathcal{S} .

Remark. The inequality (3.1) implies

$$\tilde{h}(F_{\mathbf{i}}, E_{\mathbf{i}}) \leq \delta_{\mathbf{i}} c_{\mathbf{i}} |E|, \quad (3.1')$$

where $|E|$ is the diameter of E . In the definition we may use (3.1') to replace (3.1) and obtain similar results.

3.2.2. Open set condition. F is said to satisfy the *open set condition* if there exists a family $\{V_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ of open sets such that

- (1) $F_{\mathbf{i}} \subseteq \text{cl}(V_{\mathbf{i}})$ for $\mathbf{i} \in \mathcal{I}$;
- (2) $V_{\mathbf{i}} \cap V_{\mathbf{j}} = \emptyset$ (the empty set) for $\mathbf{i}, \mathbf{j} \in \mathcal{I}$ and $\mathbf{i} \cap \mathbf{j} = \emptyset$;
- (3) there exist two positive constants a_1 and a_2 so that each $V_{\mathbf{i}}$ contains a ball of radius $a_1 c_{\mathbf{i}}$ and is contained in a ball of radius $a_2 c_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}$).

3.2.3. Lemmas. (1) Let a_1 and a_2 be two positive constants and $r > 0$. Suppose $\{V_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ is a family of disjoint open sets. If each $V_{\mathbf{i}}$ contains a ball of radius $a_1 r$ and is contained in a ball of radius $a_2 r$ ($\mathbf{i} \in \mathcal{I}$), then any closed ball B of radius r meets at most $(1 + 2a_2)^n a_1^{-n}$ of the closures $\text{cl}(V_{\mathbf{i}})$ (see [35, Lemma 9.2] or [53, Lemma 5.3(a)]).

(2) Let s be the similarity dimension $\dim_S E$ of \mathcal{S} or E , i.e., a unique solution of equation

$$\sum_{i=1}^m c_i^s = 1.$$

Define

$$\hat{\mu}(\mathbf{i}) = c_{\mathbf{i}}^s$$

for $\mathbf{i} \in \mathcal{I}$. Then $\hat{\mu}(\mathbf{i})$ can be expanded into a measure or a mass distribution on \mathcal{I}_{∞} with $\hat{\mu}(\mathcal{I}_{\infty}) = 1$.

For $A \subseteq \mathbb{R}^n$, let

$$I_A := \{\alpha \in \mathcal{I}_{\infty} : x_{\alpha} \in A \cap F\},$$

where $\{x_{\alpha}\} := \bigcap_{\mathbf{i} \ni \alpha} F_{\mathbf{i}}$, and

$$\mu(A) := \hat{\mu}(I_A).$$

Then μ is a (an outer) measure on F , i.e.,

- (i) $\mu(\emptyset) = 0$ and $\mu(A) \geq 0$ for $A \subseteq \mathbb{R}^n$;
- (ii) $\mu(A) \leq \mu(B)$ if $A \subseteq B$;
- (iii) If $A = \bigcup_{i=1}^{+\infty} A_i$ then

$$\mu(A) \leq \sum_{i=1}^{+\infty} \mu(A_i).$$

(see [34, Section 1.3], [35, the proof of Theorem 9.3] and [109, §3.2]).

(3) **Mass distribution principle** (see [35, Section 4.1] and [87, Theorem I]). Suppose that μ is a mass distribution on F (a measure on F satisfying $0 < \mu(F) < +\infty$) and for some positive constants s , c and ε we have

$$\mu(U) \leq c|U|^s$$

for any subset U of F with $|U| \leq \varepsilon$. Then

$$\mathcal{H}^s(F) \geq c^{-1}\mu(F)$$

and

$$s \leq \dim_H F \leq \underline{\dim}_B F \leq \overline{\dim}_B F,$$

where $\mathcal{H}^s(F)$ denotes s -dimensional Hausdorff measure of F and, $\dim_H F$, $\underline{\dim}_B F$ and $\overline{\dim}_B F$ denote the Hausdorff dimension, lower and upper box dimensions of F respectively (see [35, Chapters 3 and 4]). \square

3.2.4. Theorem. *If F is a δ -quasi-self-similar set with IFSS \mathcal{S} satisfying the open set condition (3.2.2), then*

$$\dim_H F = \dim_B F = s,$$

where s is the similarity dimension of \mathcal{S} (or F), and $0 < \mathcal{H}^s(F) < +\infty$.

Proof. We follow a normal method introduced by P. A. P. Moran ([87]) (refer to [35, Section 9.2] and [53, Section 5]).

From (3.1') it follows that

$$\begin{aligned} |F_{\mathbf{i}}| &\leq |E_{\mathbf{i}}| + 2\delta_{\mathbf{i}}c_{\mathbf{i}}|E| \\ &\leq (1 + 2\delta)c_{\mathbf{i}}|E| \leq (1 + 2\delta)c_{\max}^k|E| \end{aligned}$$

by Proposition 2.6.3(2), where $c_{\max} = \max\{c_i : i = 1, \dots, m\}$. Hence given $\varepsilon > 0$ there is a $k \in \mathbb{P}$ such that $|F_{\mathbf{i}}| \leq \varepsilon$ for all $\mathbf{i} \in \mathcal{I}_k$. Since

$$F = \bigcup_{\mathbf{i} \in \mathcal{I}_k} F_{\mathbf{i}},$$

we have

$$\begin{aligned} \mathcal{H}_{\varepsilon}^s(F) &\leq \sum_{\mathbf{i} \in \mathcal{I}_k} |F_{\mathbf{i}}|^s \\ &\leq (1 + 2\delta)^s |E|^s \sum_{\mathbf{i} \in \mathcal{I}_k} c_{\mathbf{i}}^s = (1 + 2\delta)^s |E|^s. \end{aligned}$$

Therefore $\mathcal{H}^s(F) \leq (1 + 2\delta)^s |E|^s$.

Now let $B = B(r)$ be a closed ball of radius $r > 0$. For any $\alpha \in \mathcal{I}_{\infty}$ choose the smallest k such that $c_{\mathbf{i}} \leq r$, where $\mathbf{i} = \alpha|_k$. Then $c_{\min}r < c_{\mathbf{i}}$, where $c_{\min} := \min\{c_i : i = 1, \dots, m\}$. Let $I(r)$ denote the set of all such \mathbf{i} . Then by the open set condition (3.2.2),

$$F = \bigcup_{\mathbf{i} \in I(r)} F_{\mathbf{i}} \subseteq \bigcup_{\mathbf{i} \in I(r)} \text{cl}(V_{\mathbf{i}}),$$

where each $V_{\mathbf{i}}$ ($\mathbf{i} \in I(r)$) contains a ball of radius $a_1 c_{\min} r$ and is contained in a ball of radius $a_2 r$. Let $I^*(r) = \{\mathbf{i} \in I(r) : B \cap V_{\mathbf{i}} \neq \emptyset\}$. Then

$$\#(I^*(r)) \leq a = (1 + 2a_2)^n a_1^{-n} c_{\min}^{-n}$$

by Lemma 3.2.3(1), where $\#I$ denotes the number of elements in I , and

$$I_{B \cap F} \subseteq \bigcup_{\mathbf{i} \in I^*(r)} \mathbf{i}.$$

Therefore

$$\begin{aligned}\mu(B) &= \mu(B \cap F) = \hat{\mu}(I_{B \cap F}) \leq \hat{\mu}\left(\bigcup_{\mathbf{i} \in I^*(r)} \mathbf{i}\right) \\ &\leq \sum_{\mathbf{i} \in I^*(r)} \hat{\mu}(\mathbf{i}) = \sum_{\mathbf{i} \in I^*(r)} c_{\mathbf{i}}^s \leq ar^s.\end{aligned}$$

For a subset U of \mathbb{R}^n let $B = B(r)$ be a closed ball of radius $r = |U|$ centered at a point of U . Then $U \subseteq B$, and consequently

$$\mu(U) \leq \mu(B) \leq ar^s = a|U|^s.$$

By Lemma 3.2.3(3) we obtain

$$\mathcal{H}^s(F) \geq a^{-1}\mu(F) = a^{-1}.$$

Let $q(r) = \sharp I(r)$. Then

$$q(r)c_{\min}^s r^s \leq \sum_{\mathbf{i} \in I(r)} c_{\mathbf{i}}^s = 1.$$

Thus $q(r) \leq c_{\min}^{-s} r^{-s}$. From (3.1') we deduce

$$\begin{aligned}|F_{\mathbf{i}}| &\leq |E_{\mathbf{i}}| + 2\delta_{\mathbf{i}}c_{\mathbf{i}}|E| \\ &\leq (1 + 2\delta_{\mathbf{i}})c_{\mathbf{i}}|E| \leq (1 + 2\delta)|E|r = br,\end{aligned}$$

where $b = (1 + 2\delta)|E|$ is a positive constant. So finally

$$\begin{aligned}\overline{\dim}_B F &= \limsup_{r \rightarrow 0^+} \frac{\log N(br)}{-\log(br)} \leq \limsup_{r \rightarrow 0^+} \frac{\log q(br)}{-\log(br)} \\ &\leq \limsup_{r \rightarrow 0^+} \frac{\log(c_{\min}^{-s} b^{-s} r^{-s})}{-\log(br)} = s,\end{aligned}$$

where $N(r)$ denotes the smallest number of sets of diameter at most r which cover F . \square

3.2.5. Corollary. *Suppose F is a δ -quasi-self-similar set with IFSS \mathcal{S} satisfying the following condition: There exists $\varepsilon_0 > 0$ such that*

$$d(F_{\mathbf{i}}, F_{\mathbf{j}}) \geq \varepsilon_0 c_{\mathbf{i}|_k} \tag{3.2}$$

for any pair \mathbf{i} and \mathbf{j} satisfying $\mathbf{i} \cap \mathbf{j} \neq \emptyset$, $\mathbf{i}|_k = \mathbf{j}|_k$ but $i_{k+1} \neq j_{k+1}$ (k is some nonnegative integer associated with \mathbf{i} and \mathbf{j}). Then

$$\dim_H F = \dim_B F = s,$$

where s is the similarity dimension of \mathcal{S} (or F), and $0 < \mathcal{H}^s(F) < +\infty$.

Proof. We only need to prove that (3.1) and (3.2) imply the open set condition.

Let

$$V_{\mathbf{i}} = \mathcal{N}\left(F_{\mathbf{i}}, \frac{1}{3}\varepsilon_0 c_{\mathbf{i}}\right).$$

By (3.1) it follows that

$$|F_{\mathbf{i}}| \leq (1 + 2\delta)|E|c_{\mathbf{i}}.$$

Hence

$$|V_{\mathbf{i}}| \leq |F_{\mathbf{i}}| + \frac{2}{3}\varepsilon_0 c_{\mathbf{i}} \leq \left[(1 + 2\delta)|E| + \frac{2}{3}\varepsilon_0\right] c_{\mathbf{i}}.$$

Let $a_1 = \frac{1}{3}\varepsilon_0$ and $a_2 = (1 + 2\delta)|E| + \frac{2}{3}\varepsilon_0$. Then

$$B_1(a_1 c_{\mathbf{i}}) \subseteq V_{\mathbf{i}} \subseteq B_2(a_2 c_{\mathbf{i}}),$$

where $B_1(a_1 c_{\mathbf{i}})$ and $B_2(a_2 c_{\mathbf{i}})$ are balls of radii $a_1 c_{\mathbf{i}}$ and $a_2 c_{\mathbf{i}}$ respectively. Denote $\mathbf{k} = \mathbf{i}|_k$ in (3.2). Then

$$\begin{aligned} d(V_{\mathbf{i}}, V_{\mathbf{j}}) &\geq d(F_{\mathbf{i}}, F_{\mathbf{j}}) - \frac{1}{3}\varepsilon_0 c_{\mathbf{i}} - \frac{1}{3}\varepsilon_0 c_{\mathbf{j}} \\ &\geq \varepsilon_0 c_{\mathbf{k}} - \frac{1}{3}\varepsilon_0 c_{\mathbf{k}} - \frac{1}{3}\varepsilon_0 c_{\mathbf{k}} = \frac{1}{3}\varepsilon_0 c_{\mathbf{k}} > 0. \end{aligned}$$

Hence $V_{\mathbf{i}} \cap V_{\mathbf{j}} = \emptyset$. □

3.2.6. Remark. We may follow an ordinary way below to construct a compact set (fractal) in \mathbb{R}^n and its structure system.

Suppose $\{G_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ is a family of nonempty compact sets in \mathbb{R}^n such that $G_{\mathbf{i}} \subseteq G_{\mathbf{j}}$ if $\mathbf{i} \subseteq \mathbf{j}$ and $|G_{\mathbf{i}}| \rightarrow 0$ ($|\mathbf{i}| \rightarrow +\infty$). Then for $\alpha \in \mathcal{I}_{\infty}$ the set $\bigcap_{\alpha \in \mathbf{i}} G_{\mathbf{i}}$ is a singleton, whose member is denoted x_{α} . Let

$$G^{(p)}(\mathbf{i}) := \bigcup_{\mathbf{j} \subseteq \mathbf{i}, |\mathbf{j}|=p} G_{\mathbf{j}}$$

for $p \geq |\mathbf{i}|$ and

$$F_{\mathbf{i}} := \bigcap_{p=|\mathbf{i}|}^{\infty} G^{(p)}(\mathbf{i}).$$

Then $G^{(p)}(\mathbf{i})$ ($p \geq |\mathbf{i}|$) are also nonempty compact sets and $G^{(p)}(\mathbf{i}) \subseteq G^{(q)}(\mathbf{i})$ if $p \geq q$, hence $F_{\mathbf{i}}$ is a nonempty compact set. Now we have constructed a nonempty compact set $F = F_{\mathbf{0}}$.

Proposition. $F_{\mathbf{i}} = \text{cl}(X_{\mathbf{i}})$, where $X_{\mathbf{i}} := \{x_{\alpha} : \alpha \in \mathbf{i}\}$.

Proof. Obviously $X_{\mathbf{i}} \subseteq F_{\mathbf{i}}$, thus $\text{cl}(X_{\mathbf{i}}) \subseteq F_{\mathbf{i}}$.

If $x \in F_{\mathbf{i}}$ then $x \in G^{(p)}(\mathbf{i})$ for all $p \geq |\mathbf{i}|$. Hence for each $p \geq |\mathbf{i}|$ there exists $\mathbf{j} \subseteq \mathbf{i}$ so that $|\mathbf{j}| = p$ and $x \in G_{\mathbf{j}}$. Therefore for any given neighborhood $N(x)$ of x we can find \mathbf{j} ($|\mathbf{j}| \geq |\mathbf{i}|$) such that $G_{\mathbf{j}} \subseteq N(x)$. Now we have $x_{\beta} \in N(x)$ for $\beta \in \mathbf{j} \subseteq \mathbf{i}$. \square

Corollary. $F_{\mathbf{i}} = \bigcup_{i=1}^m F_{\mathbf{i}i}$.

Proof. It is easy to see that

$$X_{\mathbf{i}} = \bigcup_{i=1}^m X_{\mathbf{i}i}. \quad \square$$

By the corollary above we know that $\mathcal{F} = \{F_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ is a structure system of F .

If $\{H_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ is a family of nonempty compact sets then we can let $G_{\mathbf{i}} := \bigcup_{\mathbf{j} \subseteq \mathbf{i}} H_{\mathbf{j}}$. If moreover $|G_{\mathbf{i}}| \rightarrow 0$ ($|\mathbf{i}| \rightarrow +\infty$) then we return to the above steps to construct a nonempty compact set (fractal) and its structure system.

One perhaps more useful way to perturb a self-similar set (and other similar structures) will be introduced in some concrete examples below.

3.2.7. Examples.

(1) Perturbation of the Cantor set. Let \mathcal{C} denote Cantor's ternary set, which is the invariant set of $S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ in \mathbb{R} . Let $\mathcal{C}_{\mathbf{i}} := S_{\mathbf{i}}(\mathcal{C})$.

(i) Given two families $\{a_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}(2)\}$ and $\{b_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}(2)\}$ of real numbers satisfying

$$a_0 \leq a_{\mathbf{i}} < b_{\mathbf{i}} \leq b_0, \quad (3.3)$$

where a_0 and b_0 are two fixed real numbers such that $a_0 < b_0$, we assume $H_{\mathbf{i}} = S_{\mathbf{i}}([a_{\mathbf{i}}, b_{\mathbf{i}}])$ and $G_{\mathbf{i}} = \bigcup_{\mathbf{j} \subseteq \mathbf{i}} H_{\mathbf{j}}$.

Following Remark 3.2.6 we obtain a nonempty compact set $F = F_0$ and its structure system $\mathcal{F} = \{F_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}(2)\}$. It is easy to see that

$$\tilde{h}(F_{\mathbf{i}}, \mathcal{C}_{\mathbf{i}}) \leq \frac{\delta_0}{3^k} \frac{1}{2},$$

where $\delta_0 = \max\{|b_0 - a_0 - 1|, 1\}$ and $k = |\mathbf{i}|$. Thus F is a δ_0 -perturbation of \mathcal{C} , denoted $\mathcal{C}\{a_{\mathbf{i}}, b_{\mathbf{i}}\}$. Specially we choose $a_{\mathbf{i}} = 0$ and $\mathcal{C}\{a_{\mathbf{i}}, b_{\mathbf{i}}\}$ is written by $\mathcal{C}\{b_{\mathbf{i}}\}$. Then we may let $a_0 = 0$. Assume $b_0 < 2$. We deduce

$$d(G_{\mathbf{i}}, G_{\mathbf{j}}) \geq \frac{1}{3^k} d(S_{\mathbf{i}}([0, b_0]), S_{\mathbf{j}}([0, b_0])) \geq \frac{\varepsilon_0}{3^k}$$

for $\mathbf{i}, \mathbf{j} \in \mathcal{I}(2)$ and $\mathbf{i} \cap \mathbf{j} = \emptyset$, where $\varepsilon_0 = \min \left\{ \frac{2}{3} - \frac{b_0}{3}, \frac{2}{9} \right\} > 0$, $\mathbf{i}|_k = \mathbf{j}|_k$ but $i_{k+1} \neq j_{k+1}$. By Corollary 3.2.5 we have

$$\dim_H \mathcal{C}\{b_{\mathbf{i}}\} = \dim_B \mathcal{C}\{b_{\mathbf{i}}\} = s,$$

where $s = \log 2 / \log 3$, and $0 < \mathcal{H}^s(\mathcal{C}\{b_{\mathbf{i}}\}) < +\infty$ ($0 < b_0 < 2$).

We note that $\mathcal{C}\{a_{\mathbf{i}}, b_{\mathbf{i}}\}$ is also a Moran set (see [110]).

(ii) Now let us consider the perturbation of \mathcal{C} in \mathbb{R}^2 .

If we translate, rotate, stretch or contract each $\mathcal{C}_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}(2)$) on a reasonable small scale in \mathbb{R}^2 we may get a quasi-self-similar set. For example, let us rotate each $\mathcal{C}_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}(2)$) to perturb \mathcal{C} in \mathbb{R}^2 .

Let $L_{\mathbf{i}} = S_{\mathbf{i}}([0, 1]) \times \{0\}$. At first, we rotate each L_i ($i = 1, 2$) around some point of itself to get a line segment, denoted C_i ($i = 1, 2$). Thus each $L_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}(2)$, $|\mathbf{i}| \geq 2$) is moved to another position. Let $L'_{\mathbf{i}}$ denote $L_{\mathbf{i}}$ in the new position ($\mathbf{i} \in \mathcal{I}(2)$). Note $L'_{\mathbf{i}} = C_{\mathbf{i}}$ ($|\mathbf{i}| = 1$). Then we rotate each $L'_{\mathbf{i}}$ ($|\mathbf{i}| = 2$) around some point of itself to get a line segment $C_{\mathbf{i}}$ ($|\mathbf{i}| = 2$). In this way we may get all $C_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}(2)$). Let

$$C_{\mathbf{i}}^{(p)} = \bigcup_{\mathbf{j} \subseteq \mathbf{i}, |\mathbf{j}|=|\mathbf{i}|+p} C_{\mathbf{j}} \quad (p = 0, 1, 2, \dots).$$

Then

$$\begin{aligned} C_{\mathbf{i}}^{(p+1)} &= \bigcup_{i=1}^2 C_{\mathbf{i}i}^{(p)}, \\ h(C_{\mathbf{i}}^{(p)}, C_{\mathbf{i}}^{(q)}) &= h \left(\bigcup_{\mathbf{j} \subseteq \mathbf{i}, |\mathbf{j}|=|\mathbf{i}|+p} C_{\mathbf{j}}, \bigcup_{\mathbf{j} \subseteq \mathbf{i}, |\mathbf{j}|=|\mathbf{i}|+p} C_{\mathbf{j}}^{(q-p)} \right) \\ &\leq \sup \left\{ h(C_{\mathbf{j}}, C_{\mathbf{j}}^{(q-p)}) : \mathbf{j} \subseteq \mathbf{i}, |\mathbf{j}| = |\mathbf{i}| + p \right\} \\ &\leq \frac{1}{2} \frac{1}{3^{k+p}} \end{aligned} \tag{3.4}$$

if $q = p, p+1, \dots$, and

$$\tilde{h}(C_{\mathbf{i}}^{(p)}, \mathcal{C}_{\mathbf{i}}) \leq \frac{1}{2} \frac{1}{3^k}, \tag{3.5}$$

where $k = |\mathbf{i}|$. By the completeness of (\mathbb{R}^2, h) it follows that $C_{\mathbf{i}}^{(p)}$ approaches a nonempty compact set $F_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}(2)$) in \mathbb{R}^2 as $p \rightarrow +\infty$. By (3.4) we have

$$h(C_{\mathbf{i}}^{(p)}, F_{\mathbf{i}}) \leq \frac{1}{2} \frac{1}{3^{k+p}}.$$

Hence

$$\begin{aligned} h(F_{\mathbf{i}}, F_{\mathbf{i}1} \cup F_{\mathbf{i}2}) &\leq h(F_{\mathbf{i}}, C_{\mathbf{i}}^{(p+1)}) + h \left(\bigcup_{i=1}^2 C_{\mathbf{i}i}^{(p)}, \bigcup_{i=1}^2 F_{\mathbf{i}i} \right) \\ &\leq h(C_{\mathbf{i}}^{(p+1)}, F_{\mathbf{i}}) + \max \left\{ h(C_{\mathbf{i}i}^{(p)}, F_{\mathbf{i}i}) : i = 1, 2 \right\} \rightarrow 0 \quad (p \rightarrow +\infty). \end{aligned}$$

Therefore

$$F_{\mathbf{i}} = F_{\mathbf{i}1} \cup F_{\mathbf{i}2}$$

for $\mathbf{i} \in \mathcal{I}(2)$. From (3.5) we have

$$\tilde{h}(F_{\mathbf{i}}, \mathcal{C}_{\mathbf{i}}) \leq \frac{1}{2} \frac{1}{3^k} \quad (k = |\mathbf{i}|).$$

Thus $\mathcal{C}' = \mathcal{C}'_2 := F_0$ is a 1-perturbation of \mathcal{C} with a structure system $\mathcal{F} = \{F_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}(2)\}$.

Suppose that in the above procedure of constructing the structure system \mathcal{F} , the centers of rotation are restricted to parts of $C_{\mathbf{i}}$ that are line segments of radii $r_{\mathbf{i}} \leq r_0 |C_{\mathbf{i}}|$ (r_0 is some fixed positive number less than $\frac{1}{6}$) centered at centers of $C_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}$) or absolute values θ of angles of rotation of $C_{\mathbf{i}}$ are less than or equal to $\theta_0 = \frac{\pi}{2}$ (which can actually be replaced by some larger $\theta_0 < \frac{2}{3}\pi$). Then we may deduce

$$d(F_{\mathbf{i}}, F_{\mathbf{j}}) \geq \frac{\varepsilon_0}{3^k},$$

where $\varepsilon_0 = \min \left\{ \frac{1-6r_0}{3}, \frac{1}{12} \right\}$ (for $\theta_0 = \frac{\pi}{2}$) and k is a nonnegative integer such that $\mathbf{i}|_k = \mathbf{j}|_k$ but $i_{k+1} \neq j_{k+1}$. By Corollary 3.2.5 we know that

$$\dim_H \mathcal{C}' = \dim_B \mathcal{C}' = \frac{\log 2}{\log 3} \quad (3.6)$$

and

$$0 < \mathcal{H}^s(\mathcal{C}') < +\infty, \quad (3.7)$$

where $s = \log 2 / \log 3$. In fact if we do not impose the above extra limitations on the rotation of $C_{\mathbf{i}}$ ($\mathbf{i} \in \mathcal{I}(2)$), (3.6) and (3.7) may still be true often.

The preceding discussion may similarly be conducted in \mathbb{R}^3 or even in \mathbb{R}^n to get compact sets \mathcal{C}'_3 or \mathcal{C}'_n .

(2) Perturbation of the von Koch curve. Let \mathcal{K} denote the von Koch curve (see [35] and [53]), which is the invariant set of $\mathcal{S} = \{S_1, S_2, S_3, S_4\}$ in \mathbb{R}^2 , where $S_1(z) = \frac{1}{3}z$, $S_2(z) = \frac{1}{3}e^{\frac{\pi}{3}i}z + \frac{1}{3}$, $S_3(z) = \frac{1}{3}e^{-\frac{\pi}{3}i}z + \frac{1}{2} + \frac{\sqrt{3}}{6}i$ and $S_4(z) = \frac{1}{3}z + \frac{2}{3}$ ($z \in \mathbb{C} = \mathbb{R}^2$, \mathbb{C} denotes the set of all complex numbers). Let $\mathcal{K}_{\mathbf{i}} := S_{\mathbf{i}}(\mathcal{K})$.

Let $\hat{\mathcal{K}}$ denote a random von Koch curve in [35, Chapter 15]. Let $K_0 = [0, 1] \times \{0\}$, $\hat{S}_2(z) = \frac{1}{3}e^{-\frac{\pi}{3}i}z + \frac{1}{3}$ and $\hat{S}_3(z) = \frac{1}{3}e^{\frac{\pi}{3}i}z + \frac{1}{2} - \frac{\sqrt{3}}{6}i$. Suppose $T_i = S_i$ for $i = 1, 4$ and $T_i = S_i$ for $i = 2, 3$ or $T_i = \hat{S}_i$ for $i = 2, 3$. Let

$$K^{(k)} = \bigcup_{\mathbf{i} \in \mathcal{I}_k(4)} S_{\mathbf{i}}(K_0)$$

and

$$\hat{K}^{(k)} = \bigcup_{\mathbf{i} \in \mathcal{I}_k(4)} T_{\mathbf{i}}(K_0),$$

where we choose corresponding $T_{\mathbf{i}}$, which may be different in each step to construct $T_{\mathbf{i}}$. Then $K^{(k)} \rightarrow \mathcal{K}$ and $\hat{K}^{(k)} \rightarrow \hat{\mathcal{K}}$ ($k \rightarrow +\infty$) in the sense of Hausdorff metric h . At first let us note that $\hat{\mathcal{K}}$ is a $\frac{1}{3}$ -perturbation of \mathcal{K} .

Let V be the interior of the rhombus whose vertices are $0, 1, \frac{1}{2} \pm \frac{\sqrt{3}}{6}i$. It is easy to see that $\hat{\mathcal{K}}$ satisfy the open set condition 3.2.2 for $V_{\mathbf{i}} = S_{\mathbf{i}}(V)$. Therefore

$$\dim_H \hat{\mathcal{K}} = \dim_B \hat{\mathcal{K}} = \frac{\log 4}{\log 3}$$

and

$$0 < \mathcal{H}^s(\hat{\mathcal{K}}) < +\infty,$$

where $s = \log 4 / \log 3$ (cf. [35, Exercise 15.3]).

If $T_i = S_i$ for $i = 1, 4$ and $T_i = S_i$ or \hat{S}_i for $i = 2, 3$, then we obtain another random von Koch “curve” $\tilde{\mathcal{K}}$ (the limit of $\tilde{K}^{(k)} = \bigcup_{\mathbf{i} \in \mathcal{I}_k(4)} T_{\mathbf{i}}(K_0)$ in the Hausdorff metric) (we call it a *random von Koch set*), which is a $\frac{1}{3}$ -perturbation of \mathcal{K} satisfy the open set condition 3.2.2.

Generally let

$$T_{\theta,t}(z) := \frac{1}{3}e^{i\theta}z + t \quad (\theta \in \mathbb{R}, t \in \mathbb{C}).$$

Suppose $T_{\mathbf{i}} := T_{\theta_{\mathbf{i}}, t_{\mathbf{i}}}$, where $\theta_{\mathbf{i}} \in \mathbb{R}$, $|t_{\mathbf{i}}| \leq \lambda$ (λ is a fixed nonnegative real number), and $K_{\mathbf{i}} := T_{\mathbf{i}|_1} \circ T_{\mathbf{i}|_2} \circ \cdots \circ T_{\mathbf{i}|_k}(L_0)$ ($L_0 = [0, 1] \times \{0\}$ and $k = |\mathbf{i}|$). Let

$$K_{\mathbf{i}}^{(p)} = \bigcup_{\mathbf{j} \subseteq \mathbf{i}, |\mathbf{j}|=|\mathbf{i}|+p} K_{\mathbf{j}} \quad (p = 0, 1, 2, \dots).$$

Then similarly to (3.4) it follows

$$h(K_{\mathbf{i}}^{(p)}, K_{\mathbf{i}}^{(q)}) \leq \frac{\delta_1(\lambda)}{3^{k+p}} \quad (q \geq p),$$

where $\delta_1(\lambda)$ is a nonnegative real number related to λ . Let $K_{\mathbf{i}}^{(p)} \rightarrow F_{\mathbf{i}}$ ($p \rightarrow +\infty$) for $\mathbf{i} \in \mathcal{I}(4)$. We have

$$F_{\mathbf{i}} = \bigcup_{i=1}^4 F_{\mathbf{i}i}.$$

Similarly to (3.5) we can also deduce

$$\tilde{h}(K_{\mathbf{i}}^{(p)}, \mathcal{K}_{\mathbf{i}}) \leq \frac{\delta_0(\lambda)}{3^k} \frac{1}{\sqrt{3}},$$

where $\delta_0(\lambda)$ is a nonnegative real number related to λ . Letting $p \rightarrow +\infty$ we get

$$\tilde{h}(F_{\mathbf{i}}, \mathcal{K}_{\mathbf{i}}) \leq \frac{\delta_0(\lambda)}{3^k} r(\mathcal{K}).$$

Thus $\mathcal{K}(\{\theta_i\}, \{t_i\}) := F_0$ ($\theta_i \in \mathbb{R}$ and $|t_i| \leq \lambda$) is a $\delta_0(\lambda)$ -perturbation of \mathcal{K} with a structure system $\mathcal{F} = \{F_i : i \in \mathcal{I}(4)\}$.

Now suppose $T_i := T_{\theta_i, t_i, \rho_i}$, where $\theta_i \in \mathbb{R}$, $|t_i| \leq \lambda$, $\rho_i > 0$ and

$$T_{\theta, t, \rho}(z) := \frac{\rho}{3} e^{i\theta} z + t \quad (\theta \in \mathbb{R}, t \in \mathbb{C}, \rho > 0).$$

Then following the preceding procedure we still get a δ -perturbation

$$\mathcal{K}(\{\theta_i\}, \{t_i\}, \{\rho_i\})$$

of \mathcal{K} , if we put some suitable restrictive condition \mathcal{R} on $\{\rho_i\}$, e.g. we assume only finitely many $\rho_i \neq 1$, where $\delta = \delta(\lambda, \mathcal{R})$ is a nonnegative real number related to λ and \mathcal{R} .

(3) Perturbation of the Sierpiński gasket. Let \mathcal{S} denote the Sierpiński gasket (see [35]), which is the invariant set of $\mathcal{S} = \{S_1, S_2, S_3\}$ in \mathbb{R}^2 , where $S_1(z) = \frac{1}{2}z + \frac{1}{4} + \frac{\sqrt{3}}{4}i$, $S_2(z) = \frac{1}{2}z$, $S_3(z) = \frac{1}{2}z + \frac{1}{2}$ in $\mathbb{C} = \mathbb{R}^2$. Let $\mathcal{S}_i := S_i(\mathcal{S})$ ($i \in \mathcal{I}(3)$). Suppose $A_\gamma = A\left(\gamma, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ denotes the closed triangular region whose vertices are γ , 1 and $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ ($\gamma \in \mathbb{C}$). Suppose $\Gamma = \{\gamma_i : i \in \mathcal{I}(4)\}$ satisfies that $a \leq \gamma_i < 1$ (a is a fixed real number less than 1) and that $\gamma_i \subseteq \gamma_j$ if $i \supseteq j$. Let $G_i = S_i(A_{\gamma_i})$ and let $\mathcal{S}(\Gamma)$ denote the nonempty compact set F_0 obtained by using the procedure in Remark 3.2.6. Then $\mathcal{S}(\Gamma)$ is a δ -perturbation of \mathcal{S} . If Γ satisfies some suitable condition, e.g. $0 \leq \gamma_i \leq b$, where b is a fixed nonnegative real number less than 1, then $\mathcal{S}(\Gamma)$ satisfies the open set condition 3.2.2 and thus $\mathcal{S}(\Gamma)$ is an s -set, where $s = \log 3 / \log 2$.

We may also perturb \mathcal{S} by following the way of perturbing the von Koch curve \mathcal{C} above. Generally suppose

$$T_{r, \theta, t, \sigma, \rho}(z) := r\rho e^{i\theta} \sigma(z) + t,$$

where $0 < r < 1$, $0 \leq \theta < 2\pi$, $t \in \mathbb{C}$, $\rho > 0$, $\sigma(z) = z$ or $\sigma(z) = \bar{z}$. Let

$$S_j = T_{r_j, \varphi_j, b_j, \sigma_j, 1},$$

where $0 < r_j < 1$, $0 \leq \varphi_j < 2\pi$, $b_j \in \mathbb{C}$, $\sigma_j(z) = z$ or $\sigma_j(z) = \bar{z}$ ($j = 1, \dots, m$). Let $\mathfrak{S} = \mathfrak{S}(r_j, \varphi_j, b_j, \sigma_j)$ denote the invariant set of $\mathcal{S} = \{S_1, \dots, S_m\}$. Denote

$$\mathcal{W} = \{\{\theta_i\}, \{t_i\}, \{\sigma_i\}, \{\rho_i\}\},$$

where $0 \leq \theta_i < 2\pi$, $|t_i| \leq \lambda$, $\sigma_i(z) = z$ or \bar{z} , $\{\rho_i\}$ satisfies some suitable restrictive condition \mathcal{R} (e.g. only finitely many $\rho_i \neq 1$). Suppose

$$T_i := T_{r_i, \theta_i, t_i, \sigma_i, \rho_i}(z),$$

where $\mathbf{i} = i_1 \cdots i_k \in \mathcal{I}(m)$ and $r_{\mathbf{i}} = r_{i_1} \cdots r_{i_k}$, and

$$K_{\mathbf{i}} := (T_{\mathbf{i}|_1} \circ T_{\mathbf{i}|_2} \circ \cdots \circ T_{\mathbf{i}|_k})(L_0),$$

where k is a nonnegative integer and L_0 is any fixed nonempty compact set in \mathbb{R}^2 . Let

$$K_{\mathbf{i}}^{(p)} = \bigcup_{\mathbf{j} \subseteq \mathbf{i}, |\mathbf{j}|=|\mathbf{i}|+p} K_{\mathbf{j}} \quad (p = 0, 1, 2, \dots).$$

Then

$$h(K_{\mathbf{i}}^{(p)}, K_{\mathbf{i}}^{(q)}) \leq \delta_1(\lambda, \mathcal{R}) r_{\mathbf{i}} r_{\max}^p r(\mathfrak{S}) \quad (q \geq p),$$

where $r_{\max} = \max\{r_1, \dots, r_m\}$ and $\delta_1(\lambda, \mathcal{R})$ is a nonnegative real number related to λ and \mathcal{R} . Thus $\{K_{\mathbf{i}}^{(p)}\}_{p=1}^{\infty}$ approaches a nonempty compact set in \mathbb{R}^2 as $p \rightarrow +\infty$ for each $\mathbf{i} \in \mathcal{I}(m)$, denoted $F_{\mathbf{i}}$. It is easy to see

$$F_{\mathbf{i}} = \bigcup_{i=1}^m F_{\mathbf{i}i}.$$

We can also deduce

$$\tilde{h}(F_{\mathbf{i}}, \mathfrak{S}_{\mathbf{i}}) \leq \delta_0(\lambda, \mathcal{R}) r_{\mathbf{i}} r(\mathfrak{S}),$$

where $\delta_0(\lambda, \mathcal{R})$ is a nonnegative real number related to λ and \mathcal{R} . Therefore $\mathfrak{S}(\mathcal{W}) := F_0$ is a $\delta_0(\lambda, \mathcal{R})$ -perturbation of \mathfrak{S} with a structure system $\mathcal{F} = \{F_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}(m)\}$.

We may also perturb \mathcal{K} , \mathcal{S} and \mathfrak{S} to get compact sets \mathcal{K}'_n , \mathcal{S}'_n and \mathfrak{S}'_n in \mathbb{R}^n ($n \geq 2$) similarly.

3.2.8. Remark. (1) Let $P(\mathcal{E})$ denote the probability of event \mathcal{E} . Suppose

$$\begin{aligned} \mathcal{E} = \{ \mathcal{W} : \dim_H \mathfrak{S}(\mathcal{W}) = \dim_B \mathfrak{S}(\mathcal{W}) = s \\ \text{and } 0 < \mathcal{H}^s(\mathfrak{S}(\mathcal{W})) < +\infty \}, \end{aligned}$$

where r_1, \dots, r_m are fixed so that $s = \dim_S \mathfrak{S} \leq 2$ ($\sum_{j=1}^m r_j^s = 1$), $\{\rho_{\mathbf{i}}\}$ satisfies some suitable restrictive condition \mathcal{R} or simply each $\rho_{\mathbf{i}} = 1$, each $\theta_{\mathbf{i}}$ ($0 \leq \theta_{\mathbf{i}} < 2\pi$), each $t_{\mathbf{i}}$ ($|t_{\mathbf{i}}| \leq \lambda$, λ is a fixed nonnegative real number) and each $\sigma_{\mathbf{i}}$ ($\sigma_{\mathbf{i}}(z) = z$ or $\sigma_{\mathbf{i}}(z) = \bar{z}$) possess some probability distributions, for simplicity we assume that they are evenly distributed. One question now arises: how much is $P(\mathcal{E})$?

Specially in Examples 3.2.7 for $\mathcal{C}\{a_{\mathbf{i}}, b_{\mathbf{i}}\}$ ($a_0 \leq a_{\mathbf{i}} < b_{\mathbf{i}} \leq b_0$), \mathcal{C}'_n , \mathcal{K}'_n and \mathcal{S}'_n , which do not satisfy the corresponding restrictive conditions that imply the open set condition, each $P(\mathcal{E})$ seems to be 1.

(2) More generally we may perturb a self-similar set of \mathbb{R}^n in $\mathbb{R}^{n'}$ ($n' \geq n$) similarly to perturbing \mathfrak{S} .

(3) Let \mathcal{S}' be a Sierpiński gasket which is obtained from any triangle by following the method of getting the Sierpiński gasket \mathcal{S} from a regular triangle. Then \mathcal{S} and \mathcal{S}' are δ -perturbations of one another. In fact we have the following more general result.

3.2.9. Proposition. *Let E and F be two invariant sets of IFSSs $\{S_i : i = 1, \dots, m\}$ and $\{T_i : i = 1, \dots, m\}$ respectively. If $\text{Lip } S_i = \text{Lip } T_i$ ($i = 1, \dots, m$), then E and F are δ -perturbations of one another.*

Proof. It is easy to see that $S_{\mathbf{i}}^{-1} \circ T_{\mathbf{i}}$ is an isometry. Let $\text{Lip } S_i = \text{Lip } T_i = c_i$ ($i = 1, \dots, m$). Since $F_{\mathbf{i}} = T_{\mathbf{i}}(F) = S_{\mathbf{i}}((S_{\mathbf{i}}^{-1} \circ T_{\mathbf{i}})(F))$ we get

$$\tilde{h}(F_{\mathbf{i}}, E_{\mathbf{i}}) = r_{\mathbf{i}} \tilde{h}(E, F) = c_{\mathbf{i}} \tilde{h}(E, F) = \alpha c_{\mathbf{i}} r(E),$$

where $\alpha = (r(E))^{-1} \tilde{h}(E, F)$. □

3.2.10. Remark. (1) We easily see that a singleton is a δ -perturbation of any self-similar set which is not a singleton. So a singleton may be considered as an extremely degenerate state of self-similar sets.

(2) Let C_{λ} be a λ -Cantor set which is determined by $S_1(x) = \frac{1}{3}x$, $S_2(x) = \frac{1}{3}x + \frac{2}{3}$ and $S_{\lambda}(x) = \frac{1}{3}x + \frac{\lambda}{3}$ ($0 \leq \lambda \leq 2$) (see [90]). Then by Proposition 3.2.9 we know that $C_{\lambda'}$ is a δ -perturbation of C_{λ} with IFSS $\{S_1, S_2, S_3\}$ ($0 \leq \lambda' \leq 2$). From [90] we see that C_{λ} ($0 \leq \lambda \leq 2$) have various Hausdorff dimensions. Note that $C_0 = C_2 = \mathbb{C}$ and $C_1 = [0, 1]$.

(3) Normally we regard a self-similar set as an invariant set of an IFSS which satisfies some kind of separation condition (cf. [53, 5.1]). Now we may consider any nonempty set F as a Δ -perturbation of any self-similar set E , where $\Delta = \{\delta_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$. However $\delta = \sup\{\delta_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ is probably equal to $+\infty$ often.

(4) In fact in (3.1) the metric \tilde{h} may be replaced by a simple quantity (the difference of diameters or radii) (refer to Proposition 2.6.3 and the proof of Theorem 3.2.4) and we still have the same result as Theorem 3.2.4. That is rough, however, for describing approximation degree of a fractal to a self-similar set. When the perturbation is relatively small, we may imagine that the quasi-self-similar set (defined in Definition 3.2.1) visually and intuitively possesses approximate self-similarity.

(5) Similarly to the preceding discussion for self-similar sets we may perturb some other kinds of fractals, e.g. a fractal with a graph-directed construction (refer to [80]).

3.3. Approximate self-similar sets. According to [53] if a compact set can be divided into a finite number of parts which are strictly similar to the whole part F then all details have been determined and each arbitrarily small part entirely reflects the whole. However, if the strict similarity is not required, the determination of the details will not exist any more. Now let us give the following definition.

3.3.1. Definition. Let $\mathcal{F} = \{F_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}(\{m_j\})\}$ (m_j is permitted to be $+\infty$) be a family of compact sets in \mathbb{R}^n satisfying

$$F_{\mathbf{i}} = \bigcup_{i=1}^{m_{k+1}} F_{\mathbf{i}i} \quad (k = |\mathbf{i}|).$$

We call \mathcal{F} a *structure system* of $F := F_{\mathbf{0}}$. Let $\Delta = \{\delta_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}(\{m_j\})\}$ be a family of nonnegative real numbers. Let $\delta := \sup\{\delta_{\mathbf{i}} : \delta_{\mathbf{i}} \in \Delta\}$.

(1) If

$$\widehat{h}(F_{\mathbf{i}}, F) \leq \delta_{\mathbf{i}}$$

for all $\mathbf{i} \in \mathcal{I}(\{m_j\})$, then F is called a Δ -*approximate self-similar set*. If $\delta < 1$, then F is also called a δ -*approximate self-similar set*.

(2) If

$$\widehat{h}(F_{\mathbf{i}i}, F_{\mathbf{i}}) \leq \delta_{\mathbf{i}i}$$

for each $\mathbf{i} \in \mathcal{I}(\{m_j\})$, then F is called a *level-by-level Δ -approximate self-similar set*. If $\delta < 1$, then F is also called a *level-by-level δ -approximate self-similar set*.

(3) If

$$\widehat{h}(F_{\mathbf{j}}, F_{\mathbf{i}}) \leq \delta_{\mathbf{j}}$$

for $\mathbf{i}, \mathbf{j} \in \mathcal{I}(\{m_j\})$ and $\mathbf{i} \supset \mathbf{j}$, then F is called a *uniformly (level-by-level) Δ -approximate self-similar set*. If $\delta < 1$, then F is also called a *uniformly (level-by-level) δ -approximate self-similar set*.

3.3.2. Remark. (1) If F is a δ -approximate self-similar set ($0 \leq \delta < \frac{1}{2}$), then F is a (uniformly) level-by-level 2δ -approximate self-similar set. If F is a uniformly level-by-level Δ -approximate self-similar set, then F is a Δ -approximate self-similar set.

(2) Obviously if $\delta_i = 0$ ($i = 1, \dots, m_1$) then the Δ -approximate self-similar set and (uniformly) level-by-level Δ -approximate self-similar set F are strictly self-similar. A compact set F in \mathbb{R}^n is strictly self-similar if and only if F is a 0-approximate self-similar set or a (uniformly) level-by-level 0-approximate self-similar set.

3.3.3. Proposition. If F is a Δ -perturbation of the self-similar set E , where $\Delta = \{\delta_{\mathbf{i}} \in [0, 1) : \mathbf{i} \in \mathcal{I}(m)\}$, then F is a $2(\Delta + \delta_{\mathbf{0}})$ -approximate self-similar set and is also a level-by-level Δ' -approximate self-similar set, where $2(\Delta + \delta_{\mathbf{0}}) = \{2(\delta_{\mathbf{i}} + \delta_{\mathbf{0}}) : \mathbf{i} \in \mathcal{I}(m)\}$ and $\Delta' = \{\delta'_{\mathbf{i}i} = \delta_{\mathbf{i}} + \delta_{\mathbf{i}i} : \mathbf{i} \in \mathcal{I}(m), i = 1, \dots, m\}$. Specially, if $\delta := \sup\{\delta_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}(m)\} < \frac{1}{4}$, then F is a (uniformly) level-by-level 4δ -approximate self-similar set.

Proof. Let $\mathcal{F} = \{F_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}(m)\}$ be a structure system of F so that

$$\widetilde{h}(F_{\mathbf{i}}, E_{\mathbf{i}}) \leq \delta_{\mathbf{i}} c_{\mathbf{i}} r(E),$$

where $E = E_0$ is the invariant set of $\mathcal{S} = \{S_i : \mathbf{i} \in \mathcal{I}(m)\}$, $c_i = \text{Lip } S_i$ and $E_i = S_i(E)$. Then

$$\tilde{h}(B, A_0) \leq \delta_i$$

where $A_0 = \frac{1}{r(E)}E$ and $B = \frac{1}{r(E)}S_i^{-1}(F_i)$. Let $B_0 = \frac{1}{r(B)}B$. Then

$$\begin{aligned} \hat{h}(F_i, E) &= \hat{h}(B, A_0) = \tilde{h}(B_0, A_0) \\ &\leq \tilde{h}(B_0, r(B) B_0) + \tilde{h}(B, A_0) \\ &\leq |r(B) - 1| + \tilde{h}(B, A_0) \\ &\leq 2\tilde{h}(B, A_0) \leq 2\delta_i \end{aligned}$$

by Proposition 2.6.3. Therefore

$$\hat{h}(F_i, F) \leq \hat{h}(F_i, E) + \hat{h}(E, F) \leq 2(\delta_i + \delta_0)$$

and

$$\hat{h}(F_i, F_j) \leq \hat{h}(F_i, E) + \hat{h}(E, F_j) \leq 2(\delta_i + \delta_j). \quad \square$$

3.3.4. Remark. If F is a δ -quasi-self-similar set but δ is great enough, then F may not be a δ' -approximate self-similar set or level-by-level δ' -approximate self-similar set for some $\delta' \in [0, 1)$.

3.3.5. Definition. Let $\mathcal{F}^l = \{F_i : \mathbf{i} \in \mathcal{I}^l(\{m_j\})\}$ ($l \in \mathbb{P}$) be a family of compact sets in \mathbb{R}^n satisfying

$$F_i = \bigcup_{i=1}^{m_{k+1}} F_{ii} \quad (k = |\mathbf{i}|),$$

where $k = 0, 1, \dots, l-1$. We call \mathcal{F}^l a *finite structure system* of $F := F_0$. Let $\Delta^l = \{\delta_i : \mathbf{i} \in \mathcal{I}^l(\{m_j\})\}$ be a family of nonnegative real numbers which are less than 1. Let $\delta := \max\{\delta_i : \delta_i \in \Delta^l\}$.

(1) If

$$\hat{h}(F_i, F) \leq \delta_i$$

for all $\mathbf{i} \in \mathcal{I}^l(\{m_j\})$, then F is called a Δ^l -approximate self-similar set of level l . F is also called a δ -approximate self-similar set of level l .

(2) If

$$\hat{h}(F_{ii}, F_i) \leq \delta_{ii}$$

for each $\mathbf{i} \in \mathcal{I}^{l-1}(\{m_j\})$ and $i = 1, \dots, m_{k+1}$ ($k = |\mathbf{i}|$), then F is called a *level-by-level Δ^l -approximate self-similar set of level l* . F is also called a *level-by-level δ -approximate self-similar set of level l* .

(3) If

$$\hat{h}(F_j, F_i) \leq \delta_j$$

for $\mathbf{i}, \mathbf{j} \in \mathcal{I}^l(\{m_j\})$ and $\mathbf{i} \supset \mathbf{j}$, then F is called a *uniformly (level-by-level) Δ^l -approximate self-similar set of level l* . F is also called a *uniformly (level-by-level) δ -approximate self-similar set of level l* .

3.3.6. Remark. In nature real objects rarely conform to Definition 3.3.1, but they may conform to Definition 3.3.5.

4 Comparison of fractals

Only by comparing can one distinguish. In Subsection 4.2 we introduce some concepts to describe fractals using shape differences by comparison. First we pose a problem which arises in our life.

4.1. Problem of shape vision error. In the real world, errors always exist, which include matters of human eyes. Here we suggest a problem concerning the ability of man's visual sense.

We consider plane figures, which entirely get inside visual fields of tested people and are not too far or too near from the tested people, ignoring minor details. All the plane figures and situations considered are as normal and simple as possible — we further make the following appointment: (i) The colors of figures are black, the background is white and the brightness of light is natural and moderate (of course, we may also consider effects of these factors). (ii) Only usual Euclidean figures, such as triangles, quadrilaterals (including rectangles, parallelograms, trapezoids), polygons, ellipses, sectors and so on, are considered. (iii) The figures are not too long and narrow and are not too small or too large. Lengths of sides of polygons have no wide differences, etc. (iv) We only compare between figures without any obvious distinctions. For example, we can consider shape differences between an regular triangle (polygon) and other triangles (polygons).

The problem is (let A and B be two plane figures considered): How much is the critical value when $\widehat{h}(A, B)$ is beyond it we can feel shapes of A and B are different and when $h(A, B)$ is below it we feel shapes of A and B are the same?

Remark. (1) The critical value may be replaced by a small critical interval.

(2) The results may be affected by specially appointed groups of tested people to a certain degree. But we assume the tested people are average.

(3) Complementary questions: Whether or not are the results affected by differences or ratios of radii (or diameters) of figures and sizes or shapes of figures? Whether or not are the results affected by distances between the figures and the tested people?

4.2. Atlases of fractals. Let $0 < \delta \leq \frac{1}{6}$. If we look at Cantor's ternary set \mathcal{C} , which is the invariant set of $\mathcal{S} = \{S_1, S_2\}$ ($S_1(x) = \frac{1}{3}x$ and $S_2(x) = \frac{1}{3}x + \frac{2}{3}$) in \mathbb{R} , we can easily find

$$\widehat{h}(\mathcal{C}_i, C_k) = \widehat{h}(\mathcal{C}, C_k) = \frac{1}{2} \frac{1}{3^{k+1}} \leq \delta,$$

where $\mathcal{C}_i = S_i(\mathcal{C})$ ($i \in \mathcal{I}(2)$) and $C_k = \mathcal{S}^k([0, 1])$, when $k \geq k_0 = \lceil -\log_3(6\delta) \rceil$, where $\lceil \alpha \rceil$ denotes the smallest integer more than or equal to α ($\alpha \in \mathbb{R}$). Thus we may consider \mathcal{C} to possess a structure or form of C_k for some $k \geq k_0$, if error δ is permitted. Now let us give the following

4.2.1. Definition. Let $\mathcal{F} = \{F_i : i \in \mathcal{I}(\{m_j\})\}$ be a structure system of a compact set (fractal) F in \mathbb{R}^n . Assume $|F_i| \rightarrow 0$ ($|i| \rightarrow +\infty$) and generally it is required that \mathcal{F} satisfies some kind of separation condition, e.g. the open set condition, etc. Suppose $\mathcal{E}^{(k)} := \{E_j^{(k)} : j = 1, \dots, p_k\}$ ($k = 0, 1, 2, \dots$) are families of compact sets in \mathbb{R}^n . Let $\Delta^{(k)} = \{\delta_j^{(k)} : j = 1, \dots, p_k\}$, where $0 < \delta_j^{(k)} < 1$ ($j = 1, \dots, p_k$; and $k = 0, 1, 2, \dots$).

(1) If

$$\widehat{h}(F_i, E_{j_i}) \leq \delta_{j_i}^{(k)}$$

for each $|i| \geq k$, where $\{j_i : i \in \mathcal{I}(\{m_j\})\} = \{1, \dots, p_k\}$, then $\mathcal{E}^{(k)}$ is called a k -level $\Delta^{(k)}$ -spline (or $\delta^{(k)}$ -spline if $\delta_1^{(k)} = \dots = \delta_{p_k}^{(k)} = \delta^{(k)}$) of \mathcal{F} (or F) (or E_1, \dots, E_{p_k} are called k -level $\Delta^{(k)}$ -splines of \mathcal{F} (or F)) and we say F possesses a k -th-level structure $\mathcal{E}^{(k)}$ (or structures E_1, \dots, E_{p_k}) of error $\Delta^{(k)}$ (or $\delta^{(k)}$ if $\delta_1^{(k)} = \dots = \delta_{p_k}^{(k)} = \delta^{(k)}$). If F possesses a 0-th-level structure $\mathcal{E}^{(0)}$ of error $\Delta^{(0)}$ (or $\delta^{(0)}$) then we also say F possesses a structure $\mathcal{E} = \mathcal{E}^{(0)}$ of error $\Delta = \Delta^{(0)}$ (or $\delta = \delta^{(0)}$) and \mathcal{E} is called a Δ -spline (or δ -spline).

(2) If for any $\delta > 0$, F possesses a k -th-level structure $\mathcal{E}^{(k)}$, consisting of p_k compact sets, of error δ , then F is said to possess a k -th-level p_k -structure. If $p_k = 1$ then F is said to possess a single structure in k -th-level. If F possesses a 0-th-level p_0 -structure then F is said to possess a p -structure ($p = p_0$). If $p = 1$ then F is said to possess a single structure.

(3) Let $\lambda > 0$. If for all $i \in \mathcal{I}(\{m_j\})$ we have $|F_i| \leq \lambda|F|$ then a k -th-level structure $\mathcal{E}^{(k)}$ of error $\Delta^{(k)}$ (or $\delta^{(k)}$) of F is called a λ -degree structure of error $\Delta = \Delta^{(k)}$ (or $\delta = \delta^{(k)}$) and a k -level Δ -spline (or δ -spline) of F is called a λ -degree k -level Δ -spline (or δ -spline).

4.2.2. Remark. (1) A self-similar set F may be considered to possess a 0-spline F . But now we make a convention that in \mathbb{R}^n the splines should be geometric patterns consisting of finite formal Euclidean figures, which are called *Euclidean patterns*. It is easy to see that a self-similar set still possesses a single structure.

(2) The splines of fractals F are not unique. A standard of searching for splines of F is trying to find splines of F which can help us to see and understand details of F approximately. A well-chosen spline of a fractal F is called an *atlas* of F .

4.2.3. Examples. (1) Let $0 < \delta \leq \frac{1}{2}$. We consider the von Koch curve \mathcal{K} , see Example 3.2.7(2). Then

$$\widehat{h}(S_{\mathbf{i}}(\mathcal{K}), K^{(k)}) = \widehat{h}(\mathcal{K}, K^{(k)}) \leq \frac{1}{2} \frac{1}{3^k} \leq \delta,$$

if $k \geq k_1 = \lceil \log_3 \frac{1}{2\delta} \rceil$, where $\lceil \alpha \rceil$ denotes the smallest integer more than or equal to α ($\alpha \in \mathbb{R}$). Hence $K^{(k)}$ is a δ -spline of \mathcal{K} for each $k \geq k_1$.

(2) Let $0 < \delta \leq \frac{1}{4}$. We consider the Sierpiński gasket \mathcal{S} , see Example 3.2.7(3). Let $A^{(k)} = \mathcal{S}^{(k)}(A_0) := \bigcup_{\mathbf{i} \in \mathcal{I}_k(3)} S_{\mathbf{i}}(A_0)$ ($k = 0, 1, 2, \dots$). Then

$$\widehat{h}(S_{\mathbf{i}}(\mathcal{S}), A^{(k)}) = \widehat{h}(\mathcal{S}, A^{(k)}) \leq \frac{1}{4} \frac{1}{2^k} \leq \delta,$$

if $k \geq k_2 = \lceil \log_2 \frac{1}{4\delta} \rceil$. Hence $A^{(k)}$ is a δ -spline of \mathcal{S} for each $k \geq k_2$.

4.2.4. Remark. (1) If F is a δ -approximate self-similar set ($0 \leq \delta < 1$), then there is a $(\delta + \varepsilon)$ -spline E of F for any $\varepsilon > 0$ satisfying $\delta + \varepsilon < 1$, where E is an Euclidean pattern. But F may not possess any single structures. And a δ -quasi-self-similar set ($\delta > 0$) may not possess any single structures either (cf. Remark 4.2.2(1)).

(2) The construction object in a graph directed construction (refer to [80]) may possess a p -structure ($p \leq n$).

In order to distinguish the simplicity and complexity of details of a fractal we give the following

4.2.5. Definition. If a compact set F possesses a structure system $\mathcal{F} = \{F_{\mathbf{i}} : \mathbf{i} \in \mathcal{I}\}$ and there exist a finite number of compact sets E_1, \dots, E_q such that for each $\mathbf{i} \in \mathcal{I}$ there exists $\mathbf{j}_{\mathbf{i}} \in \{1, \dots, q\}$ satisfying

$$\widehat{h}(F_{\mathbf{i}}, E_{\mathbf{j}_{\mathbf{i}}}) \rightarrow 0 \quad (|\mathbf{i}| \rightarrow +\infty),$$

then F is said to *approach finite structure*.

4.2.6. Remark. A self-similar set and a graph directed construction object approach finite structure, but a δ -quasi-self-similar set and a δ -approximate self-similar set may not approach finite structure.

4.2.7. Fractal indices. Let F be a compact set (fractal) in \mathbb{R}^n and let $\mathcal{F} = \{F_i : i \in \mathcal{I}\}$ be a structure system of F . Assume $\delta, \lambda > 0$ and k is a nonnegative integer.

(1) Denote

$$N(\delta, \lambda; F) := \min\{\sharp E_\delta(\lambda) : E_\delta(\lambda) \text{ is a } \lambda\text{-degree} \\ \text{structure of } F \text{ of error } \delta\}$$

($\sharp A$ denotes the cardinal number of A). Then $N(\delta, \lambda; F)$ is a decreasing function of λ . Let

$$N_\delta(F) = N(\delta, F) := \sup_{\lambda > 0} N(\delta, \lambda; F).$$

Then

$$N_\delta(F) = \lim_{\lambda \rightarrow 0^+} N(\delta, \lambda; F).$$

The faster the growth of $N_\delta(F)$ on $\frac{1}{\delta}$ is, the more complex the detail of F is; and the slower the growth is, the simpler the detail is. So we call $N_\delta(F)$ the *fractal δ -index (index function)* of F ($(N_\delta(F))^{-1}$ is called the *δ -self-similarity index (function)* of F) and call the (upper, lower) growth order of $N_\delta(F)$ on $\frac{1}{\delta}$ the (*upper, lower*) *fractal order* of F , where the *upper* and *lower growth order* are

$$\bar{\rho}(F) := \limsup_{\delta \rightarrow 0^+} \frac{N_\delta(F)}{-\log \delta} \quad \text{and} \quad \underline{\rho}(F) := \liminf_{\delta \rightarrow 0^+} \frac{N_\delta(F)}{-\log \delta}$$

respectively, and if $\bar{\rho}(F) = \underline{\rho}(F)$ then the *growth order* is $\rho(F) = \bar{\rho}(F)$.

(2) Denote

$$N_\delta^{(k)}(\mathcal{F}) := \min\{\sharp E_\delta^{(k)} : E_\delta^{(k)} \text{ is a } k\text{-th-level } \delta\text{-spline of } \mathcal{F}\}.$$

Let

$$\overline{N}_\delta(\mathcal{F}) := \limsup_{k \rightarrow +\infty} N_\delta^{(k)}(\mathcal{F}) \quad \text{and} \quad \underline{N}_\delta(\mathcal{F}) := \liminf_{k \rightarrow +\infty} N_\delta^{(k)}(\mathcal{F}).$$

Define

$$\overline{N}_\delta(F) := \min\{\overline{N}_\delta(\mathcal{F}) : \mathcal{F} \text{ is a structure system of } F\}, \\ \underline{N}_\delta(F) := \min\{\underline{N}_\delta(\mathcal{F}) : \mathcal{F} \text{ is a structure system of } F\}$$

and

$$N_\delta^{(0)}(F) := \min\{N_\delta^{(0)}(\mathcal{F}) : \mathcal{F} \text{ is a structure system of } F\},$$

which are called *upper, lower* and *whole fractal δ -indices (index functions)* of F respectively. We also call $(\overline{N}_\delta(F))^{-1}$, $(\underline{N}_\delta(F))^{-1}$ and $(N_\delta^{(0)}(F))^{-1}$ *upper, lower* and *whole δ -self-similarity indices (index functions)* of F respectively.

(3) Let $F^{(j)}$ ($j \in J$, J is an index set) be fractals in \mathbb{R}^n . Denote $F_J := \{F^{(j)} : j \in J\}$. Assume $\mathcal{F}_j := \{F_{\mathbf{i}^{(j)}}^{(j)} : \mathbf{i}^{(j)} \in \mathcal{I}^{(j)}\}$ is a structure system of $F^{(j)}$ ($j \in J$). Denote $\mathcal{F}_J := \{\mathcal{F}_j : j \in J\}$, called a *structure system* of F_J . A family $\mathcal{E}_J^{(k)} = \{E_l^{(k)} : l = 1, 2, \dots, p_k\}$ of compact sets is called a δ -spline of \mathcal{F}_J (or F_J) if

$$\widehat{h}(F_{\mathbf{i}^{(j)}}^{(j)}, E_{l(j, \mathbf{i}^{(j)})}^{(k)}) \leq \delta$$

for $|\mathbf{i}^{(j)}| \geq k$ ($\mathbf{i}^{(j)} \in \mathcal{I}^{(j)}$, $j \in J$), where

$$\{l(j, \mathbf{i}^{(j)}) = l_{\mathbf{i}^{(j)}}^{(j)} : j \in J, \mathbf{i}^{(j)} \in \mathcal{I}^{(j)}, |\mathbf{i}^{(j)}| \geq k\} = \{1, 2, \dots, p_k\}.$$

Assume p_k is the smallest one such that $\mathcal{E}_J^{(k)}$ is a δ -spline of \mathcal{F}_J . Suppose

$$\begin{aligned} & \{E_{l_t}^{(k)} : t = 1, 2, \dots, q_k; 1 \leq l_1 < l_2 < \dots < l_{q_k} \leq p_k\} \\ &= \{E_l^{(k)} : 1 \leq l \leq p_k, \text{ and for each } j \in J \\ & \quad \text{there exists } \mathbf{i}^{(j)} \in \mathcal{I}^{(j)} \text{ such that } \widehat{h}(F_{\mathbf{i}^{(j)}}^{(j)}, E_l^{(k)}) \leq \delta\} \end{aligned}$$

is a common δ -spline of \mathcal{F}_J . Denote

$$\gamma_k(\mathcal{F}_J, \delta) := \frac{q_k}{p_k}.$$

Let

$$\overline{\gamma}(\mathcal{F}_J, \delta) := \limsup_{k \rightarrow +\infty} \gamma_k(\mathcal{F}_J, \delta) \quad \text{and} \quad \underline{\gamma}(\mathcal{F}_J, \delta) := \liminf_{k \rightarrow +\infty} \gamma_k(\mathcal{F}_J, \delta).$$

Define

$$\begin{aligned} \overline{\gamma}(F_J, \delta) &:= \max\{\overline{\gamma}(\mathcal{F}_J, \delta) : \mathcal{F}_J \text{ is a structure system of } F_J\}, \\ \underline{\gamma}(F_J, \delta) &:= \max\{\underline{\gamma}(\mathcal{F}_J, \delta) : \mathcal{F}_J \text{ is a structure system of } F_J\} \end{aligned}$$

and

$$\gamma_0(F_J, \delta) := \max\{\gamma_0(\mathcal{F}_J, \delta) : \mathcal{F}_J \text{ is a structure system of } F_J\},$$

which are called *upper*, *lower* and *whole* δ -similarity indices (*similarity index functions*) of F_J (or $F^{(j)}$ ($j \in J$)) respectively. If $\overline{\gamma}(F_J, \delta) = \underline{\gamma}(F_J, \delta)$ then it is called the δ -similarity index (*similarity index function*), denoted $\gamma(F_J, \delta)$. Define

$$\begin{aligned} \overline{\gamma}(F_J) &:= \limsup_{\delta \rightarrow 0^+} \overline{\gamma}(F_J, \delta), \\ \underline{\gamma}(F_J) &:= \limsup_{\delta \rightarrow 0^+} \underline{\gamma}(F_J, \delta) \end{aligned}$$

and

$$\gamma_0(F_J) := \limsup_{\delta \rightarrow 0^+} \gamma_0(F_J, \delta),$$

which are called *upper*, *lower* and *whole similarity indices* of F_J (or $F^{(j)}$ ($j \in J$)) respectively. If the above superior limits are changed into inferior limits then they are called *upper*, *lower* and *whole strong similarity indices* of F_J (or $F^{(j)}$ ($j \in J$)) respectively.

It is obvious to see that $0 \leq \underline{\gamma}(F_J, \delta) \leq \overline{\gamma}(F_J, \delta) \leq 1$, $0 \leq \gamma_0(F_J, \delta) \leq 1$, $0 \leq \underline{\gamma}(F_J) \leq \overline{\gamma}(F_J) \leq 1$ and $0 \leq \gamma_0(F_J) \leq 1$.

4.2.8. Example. If F is a self-similar set then $N_\delta(F) = 1$. If $F = \bigcup_{i=1}^m F_i$ is the construction object in a graph directed construction, then $N_\delta(F) \leq m$, and usually when δ is small enough we have $N_\delta(F) = m$.

4.2.9. Remark. In general, computing $N_\delta(F)$, $\overline{\gamma}(F_J, \delta)$, $\underline{\gamma}(F_J, \delta)$, $\gamma_0(F_J, \delta)$, $\overline{\gamma}(F_J)$, $\underline{\gamma}(F_J)$ and $\gamma_0(F_J)$ is a very difficult job. But for some $\delta > 0$ and a structure system $\mathcal{F} = \{F_i : i \in \mathcal{I}\}$ of F , computing $\overline{\gamma}(\mathcal{F}_J, \delta)$, $\underline{\gamma}(\mathcal{F}_J, \delta)$ and $\gamma_0(\mathcal{F}_J, \delta)$ may be a piece of operable work.

4.3. Some examples and remarks. A cookie-cutter set E is a quasi-self-similar set in the sense that every small piece of E can be uniformly expanded to a standard size and then mapped quasi-isometrically back into E and that E can also be quasi-isometrically contracted to any small part of E (see [31], [34, Chapter 4] and [81]). Hence E is an s -set with $s = \dim_H E = \dim_B E$ (see [34, Corollary 4.6]). A lot of research work has been done in this aspect, for example, connecting with thermodynamic formalism, we may refer to [12], [17], [34], [93], [94] and [100], etc.

4.3.1. First let us consider an example which was given in [34, Section 4.1]: Let F be a cookie-cutter set that is an invariant set of g_1 and $g_2 : [0, 1] \rightarrow [0, 1]$, where

$$g_1(x) = \frac{1}{3}x + \frac{1}{10}x^2 \quad \text{and} \quad g_2(x) = \frac{1}{3}x + \frac{2}{3} - \frac{1}{10}x^2,$$

which is a nonlinear perturbation of Cantor's ternary set. Of course we may consider a more general case:

$$g_1(x) = \frac{1}{3}x + ax^\alpha \quad \text{and} \quad g_2(x) = \frac{1}{3}x + \frac{2}{3} - bx^\beta,$$

where a and b are two small positive numbers and $\alpha, \beta > 1$. We pose the following questions:

- (1) Does F approach finite structure?

(2) If we change g_1 and g_2 above into

$$g_1(x) = \frac{1}{3}x + \varphi(x) \quad \text{and} \quad g_2(x) = \frac{1}{3}x + \frac{2}{3} - \psi(x),$$

where φ and ψ are chosen such that the new g_1 and g_2 are also contraction maps from $[0, 1]$ to itself and the invariant set F of g_1 and g_2 approaches finite structure, then what features should φ and ψ possess? (Obviously if $\varphi(x) = \psi(x) \equiv 0$ then F becomes Cantor's ternary set, which approaches finite structure.)

4.3.2. Consider the logistic map

$$f(x) = \lambda x(1 - x),$$

where λ is a positive constant. It is an important one dimensional dynamic system, which has been deeply and systematically studied. There is a known universal constant, the Feigenbaum constant, for example (refer to [35, Section 13.2], [38] and [39], etc).

If $\lambda > 2 + \sqrt{5}$, then $f : [0, a] \cup [1-a, 1] \rightarrow [0, 1]$, where $a = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\lambda}}$, gives a cookie-cutter set, denoted E_λ (refer to [34, Section 4.1] and [35, Section 13.2]). A question similar to 4.3.1(1) may be raised:

(1) Do E_λ ($\lambda > 2 + \sqrt{5}$) approach finite structure?

When $\lambda = \lambda_\infty \approx 3.570$, the attractor E_{λ_∞} is a set of Cantor type, whose Hausdorff dimension is about 0.538 (see [35, Section 13.2]). One more similar question may be mentioned:

(2) Does E_{λ_∞} approach finite structure?

4.3.3. Iterating rational functions (generally, meromorphic functions) in the complex plane \mathbb{C} a large number of fractals can emerge. Specially a quadratic polynomial

$$f_c(z) = z^2 + c \quad (c \in \mathbb{C})$$

may bring about a colorful dynamic system, which has been one of central issues in the research of complex analytic dynamic systems, and many splendid and deep results have been found. Here we only mention a few of references: [10], [14], [15], [20], [70], [83], [84], [85], [88], [89], [99], [102], [105], [107], etc.

Let \mathcal{M} denote the Mandelbrot set and \mathcal{J}_c denote the Julia set of f_c . It is known that Julia sets of hyperbolic rational functions are quasi-self-similar in the sense mentioned above in this subsection (see [15, Theorem 8.6] and [104, p. 742]) (it is not always true for all \mathcal{J}_c , see [57]). In [106] Lei Tan obtained a kind of similarity between \mathcal{M} and \mathcal{J}_c for Misiurewicz points c .

If $|c| > \frac{5+2\sqrt{6}}{4}$, then \mathcal{J}_c is a cookie-cutter set in plane. A similar question may still be asked:

- (1) Do \mathcal{J}_c ($|c| > \frac{5+2\sqrt{6}}{4}$) approach finite structure?

Let F be \mathcal{M} , \mathcal{J}_c or a piece of them and let F_J be a family of fractals chosen from \mathcal{M} , \mathcal{J}_c or pieces of them. Now our questions are:

- (2) Find the δ -self-similarity index $(N_\delta(F))^{-1}$ of F ;
- (3) Find upper, lower and whole δ -similarity indices $\overline{\gamma}(F_J, \delta)$, $\underline{\gamma}(F_J, \delta)$ and $\gamma_0(F_J, \delta)$ and upper, lower and whole similarity indices $\overline{\gamma}(F_J)$, $\underline{\gamma}(F_J)$ and $\gamma_0(F_J)$ of F_J .

The same work may be done in 4.3.1 and 4.3.2.

5 Notes on tilings, patterns, packings and crystals

The research on tilings, patterns and packings has been extensively carried out, which has a long history and was also motivated by Hilbert's 18th problem and theories about the structure of solid matter. From [19], [24] and [44] we can see numerous problems on the areas still remain open. For tilings and patterns we may refer to [49], [50] and [95], etc. For packings we may refer to [16], [23], [40], [91] and [113], etc. For (mathematical) crystallography we may refer to [2], [64], [97] and [98], etc. For some tilings the tiles may be fractals, see e.g. [4] and [73]. The rigidity of tilings was considered in [60]. For self-similar and self-affine tilings we refer to e.g. [58], [59], [61], [62], [63], [65], [66], [67], [101], [103] and [108], etc.

5.1. Tilings with quasi-prototiles. It is known that a *tiling* (of \mathbb{R}^n) is a countable family of closed sets (usually *bodies*, which are bounded and are closures of their interiors) in \mathbb{R}^n , whose union is the whole space and whose interiors are pairwise disjoint (refer to [50] and [95]). A *tile* is an element of a tiling. A body T *tiles* \mathbb{R}^n means that there is a tiling of \mathbb{R}^n whose tiles are congruent copies of T .

5.1.1. Tilings with quasi-prototiles. Let $\mathcal{T} = \{T_i : i \in I\}$ be a tiling. Let $\mathcal{P} = \{P_j : j \in J\}$ be a set of at most countable nonempty compact subsets (usually bodies) of \mathbb{R}^n and $\{\delta_j\}_{j \in J}$ be a (countable or finite) sequence of nonnegative real numbers. Suppose for each $i \in I$ there exists $j \in J$ such that

$$\tilde{h}(T_i, P_j) \leq \delta_j r(P_j),$$

where we assume every P_j ($j \in J$) is taken at least once or more strictly any P_j ($j \in J$) can not be lost. Then we call \mathcal{T} a *tiling with $\{\delta_j\}$ -quasi-prototile*

types P_j ($j \in J$) or with quasi-prototile types $\langle P_j, \delta_j \rangle$ ($j \in J$). We may also say that \mathcal{T} is a $\{\delta_j\}$ -quasi-tiling with prototile types P_j ($j \in J$) or prototile types P_j ($j \in J$) admits a $\{\delta_j\}$ -quasi-tiling \mathcal{T} . A tiling with a single quasi-prototile type is called a *monohedral quasi-tiling*. A tiling with k quasi-prototile types is called a *k-hedral quasi-tiling*. If $\delta_j = \delta$ for all $j \in J$, then \mathcal{T} is called a *tiling with δ -quasi-prototile type (set) \mathcal{P}* or *δ -quasi-prototile types P_j ($j \in J$)* or is called a *δ -quasi-tiling with prototile types P_j ($j \in J$)*. If $P_j \in \mathcal{T}$ ($j \in J$) then we also call (quasi-)prototile types P_j ($j \in J$) (*quasi-*)prototiles. Usually we may not distinguish between (quasi-)prototile types and (quasi-)prototiles and usually we assume the cardinal number $\sharp J = k$ of J is finite.

5.1.2. Quasi-symmetry groups of tilings. Let \mathcal{T} be a tiling and $\lambda \geq 0$. If the isometry φ of \mathbb{R}^n satisfies that for any $i \in I$ there exist i' and $i'' \in I$ such that

$$h(\varphi(T_i), T_{i'}) \leq \lambda r(T_i)$$

and

$$h(T_i, \varphi(T_{i''})) \leq \lambda r(T_i),$$

then φ is called a λ -quasi-symmetry of \mathcal{T} . A group \mathcal{G} consisting of λ -quasi-symmetries of \mathcal{T} is called a λ -quasi-symmetry group of \mathcal{T} . Note that a 0-quasi-symmetry of \mathcal{T} is a symmetry of \mathcal{T} and a 0-quasi-symmetry group of \mathcal{T} is a symmetry group of \mathcal{T} .

5.1.3. Quasi-self-similar tilings (cf. [58] and [95]). Recall that a *hierarchical tiling* is a tiling whose tiles (called *level-0* tiles) can be composed into larger tiles, called *level-1* tiles, whose level-1 tiles can be composed into *level-2* tiles, and so on ad infinitum (see [95, p. 66]).

(1) Let $\mathcal{Q} = \{Q_i : i \in I\}$ be a set of at most countable nonempty compact subsets (usually bodies) of \mathbb{R}^n and let $\lambda_{ij} \geq 0$ ($i \in I$ and $j \in \mathbb{N}$) (\mathbb{N} denotes the set of all nonnegative integers). We define a $\{\lambda_{ij}\}$ -quasi-self-similar tiling with a quasi-prototype (set) \mathcal{Q} as a hierarchical tiling \mathcal{T} satisfying that for any level- j tile T_j of \mathcal{T} there exists a quasi-prototype $Q_i \in \mathcal{Q}$ such that

$$\widehat{h}(T_j, Q_i) \leq \lambda_{ij}.$$

We also call \mathcal{T} a *quasi-self-similar tiling with a $\{\lambda_{ij}\}$ -quasi-prototype \mathcal{Q}* . If $\lambda_{ij} = \lambda$ for all $i \in I$ and $j \in \mathbb{N}$, then we say that \mathcal{T} is a λ -quasi-self-similar tiling with a quasi-prototype \mathcal{Q} or a *quasi-self-similar tiling with a λ -quasi-prototype \mathcal{Q}* . Usually we assume $\sharp I = k$ is finite.

(2) Let $\lambda_j \geq 0$ ($j \in \mathbb{P}$). If a hierarchical tiling \mathcal{T} satisfies that for any level- j tile T_j ($j \in \mathbb{P}$) and each level- $(j-1)$ tile $T_{j-1} \subseteq T_j$ such that

$$\widehat{h}(T_j, T_{j-1}) \leq \lambda_j \quad (j \in \mathbb{P}),$$

then \mathcal{T} is called a *level-by-level $\{\lambda_j\}$ -quasi-self-similar tiling*. If $\lambda_j = \lambda$ for all $j \in \mathbb{P}$, then we say that \mathcal{T} is a *level-by-level λ -quasi-self-similar tiling*.

(3) Let $\mathcal{Q} = \{Q_i : i \in I\}$ be a set of at most countable nonempty compact subsets (usually bodies) of \mathbb{R}^n . Let $\lambda_{ij} \geq 0$, $c_{ij} \geq 1$ ($i \in I$ and $j \in \mathbb{P}$) and $\sup\{c_{ij} : j \in \mathbb{P}\} > 1$ ($i \in I$). If a hierarchical tiling \mathcal{T} satisfies that for any level- j tile T_j ($j \in \mathbb{P}$) there exists $Q_i \in \mathcal{Q}$ and a similitude φ_{ij} of lipschitz constant c_{ij} such that

$$\widetilde{h}(T_j, \varphi_{ij}(Q_i)) \leq \lambda_{ij} c_{ij} r(Q_i),$$

then \mathcal{T} is called a *$\{\lambda_{ij}\}$ -quasi-self-similar tiling with a quasi-prototype \mathcal{Q}* or a *quasi-self-similar tiling with a $\{\lambda_{ij}\}$ -quasi-prototype \mathcal{Q} of ratio $\{c_{ij}\}$* (λ -quasi-self-similar tiling if $\lambda_{ij} = \lambda$ for all $i \in I$ and $j \in \mathbb{P}$) (of ratio $\{c_i\}$ if $c_{ij} = c_i^j$ for all $i \in I$ and $j \in \mathbb{P}$, or of ratio c if furthermore $c_i = c$ for all $i \in I$). Usually we assume $\sharp I = k$ is finite.

(4) If there is a quasi-self-similar tiling \mathcal{T} with a single $\{\lambda_j\}$ -quasi-prototype Q (of ratio $\{c_j\}$), then Q is called a *rep $\{\lambda_j\}$ -quasi-tile type* (k -rep $\{\lambda_j\}$ -quasi-tile type if every level- j tile consists of k level- $(j-1)$ tiles, $j \in \mathbb{N}$ and $k \geq 2$ is a natural number independent of $j \in \mathbb{N}$) (of ratio $\{c_j\}$). If $Q \in \mathcal{T}$ then the (k) -rep $\{\lambda_j\}$ -quasi-tile type Q is also called a (k) -rep $\{\lambda_j\}$ -quasi-tile. Usually we do not distinguish between these two notions. Similar statements can be made for the case $\lambda_j = \lambda$ for all $j \in \mathbb{N}$.

5.1.4. Example. Let

$$S_{jk} := \{(x, y) : j \leq x \leq j+1, k \leq y \leq k+1\},$$

where $j, k \in \mathbb{Z}$ (\mathbb{Z} denotes the set of all integers). Then $\mathcal{S} = \{S_{jk} : j, k \in \mathbb{Z}\}$ is a tiling with a prototile S_{00} , which is also considered as the tiling from partitioning the plane \mathbb{R}^2 into squares by lines L'_k and L''_j ($j, k \in \mathbb{Z}$), where L'_t is line $y = t$ and L''_t is line $x = t$ ($t \in \mathbb{R}$).

Let $0 \leq \delta < \frac{1}{2}$. Assume that C'_k is a Jordan curve between $L'_{k-\delta}$ and $L'_{k+\delta}$ and C''_j is a Jordan curve between $L''_{j-\delta}$ and $L''_{j+\delta}$ such that $C'_k \cap C''_j$ is a singleton for each pair $j, k \in \mathbb{Z}$. Let \mathcal{T} be a tiling from partitioning \mathbb{R}^2 into pieces by curves C'_k and C''_j ($j, k \in \mathbb{Z}$). Then \mathcal{T} is a monohedral tiling with a δ -quasi-prototile S_{00} . It is easy to see that translations $\tau(j, k) : z \mapsto z + j + ki$ ($j, k \in \mathbb{Z}$) and rotations $\sigma_m : z \mapsto e^{\frac{m\pi i}{2}} z$ ($m = 0, 1, 2, 3$) ($z \in \mathbb{C} = \mathbb{R}^2$) are λ -quasi-symmetries of \mathcal{T} and the transformation group \mathcal{G} generated by

$$\{\sigma_m, \tau(j, k) : m = 0, 1, 2, 3; j, k \in \mathbb{Z}\}$$

is a λ -quasi-symmetry group of \mathcal{T} , where $\lambda = \frac{4\delta}{1-2\delta}$. We also see that \mathcal{T} is a $\{\frac{\delta}{2^{j-1}}\}$ -quasi-self-similar tiling with quasi-prototype S_{00} of ratio 2 and any closed plane domain P containing square $S' = \{(x, y) : \delta \leq x \leq 1 - \delta, \delta \leq y \leq 1 - \delta\}$ and contained in square $S'' = \{(x, y) : -\delta \leq x \leq 1 + \delta, -\delta \leq y \leq 1 + \delta\}$ is a 4-rep $\{\lambda_j\}$ -quasi-tile ($\lambda_j = \frac{(2+2^{1-j})\delta}{1-2\delta}$) or 4-rep λ -quasi-tile ($\lambda = \frac{4\delta}{1-2\delta}$) of ratio 2.

The above example can be easily generalized to cases of space and k -hedral quasi-tilings.

5.1.5. Quasi-isohedral tilings and quasi-anisohedral tiles (cf. [95]).

Let $\delta, \delta_0, \delta_1, \delta_2 \geq 0$.

(1) *δ -quasi-transitive action.* We say that a transformation group \mathcal{G} acts *δ -quasi-transitively* on a family $\mathcal{A} = \{A_i : i \in I\}$ of subsets of \mathbb{R}^n if given any $i, j \in I$ there exists $g_{ij} \in \mathcal{G}$ such that

$$h(A_i, g_{ij}(A_j)) \leq \delta,$$

and on the other hand, for any $A \in \mathcal{A}$ and any $g \in \mathcal{G}$ there exists $B \in \mathcal{A}$ such that

$$h(B, g(A)) \leq \delta.$$

If \mathcal{G} acts δ -quasi-transitively on \mathcal{A} , then \mathcal{A} is called a *δ -quasi-orbit* of \mathcal{G} .

(2) A (δ_1, δ_2) -*quasi-isohedral tiling* is a tiling whose δ_1 -quasi-symmetry group acts δ_2 -quasi-transitively on its tiles. Hence an isohedral tiling is a $(0, 0)$ -quasi-isohedral tiling.

(3) A $(\delta_0, \delta_1, \delta_2)$ -*quasi-anisohedral tile (type)* is a prototile (type) that admits at least one monohedral δ_0 -quasi-tiling but no (δ_1, δ_2) -quasi-isohedral tilings.

5.1.6. Remark. (1) In 5.1.1, 5.1.2 and 5.1.3 we may replace $r(\cdot)$ by $|\cdot|$ to get similar definitions.

(2) We may similarly define a quasi-tiling of $W \subseteq \mathbb{R}^n$ and a quasi-symmetry φ (quasi-symmetry group) of a tiling of W .

(3) Since a quasi-tiling is also a tiling, some related results for a tiling, such as the normality lemma (see [50, 3.2.2] and [95, p.55]), still hold for a quasi-tiling (under suitable conditions).

5.1.7. Some related questions. We may consider *extensions of some relative classical results to the case of quasi-tilings*. Below we still pose a few of questions, some of which are related to Hilbert's eighteenth problem (see [19, Section 4.1], [49], [51] and [97, Sections 1.5 and 1.7], etc).

We remark here that there has existed a relative question mentioned in [50, p.497] (see also [19, Problem 4 in Section 4.1]).

Let $\delta, \delta_0, \delta_1, \delta_2 \geq 0$.

(1) If there exists a $(\delta_0, \delta_1, \delta_2)$ -quasi-anisohedral tile, then what is the relation of δ_0 , δ_1 and δ_2 ?

(2) According to [49, p.955 and p.956] it is more hopeless to determine all δ -quasi-prototiles of monohedral tilings in \mathbb{R}^n for $\delta > 0$. However, because of this reason, we may find more δ -quasi-prototiles of monohedral tilings if δ is greater (assume we do not require a whole list is shown). Perhaps we can find a large majority of δ -quasi-prototiles of monohedral tilings with the aid of computers.

As $\delta \geq 0$ gets smaller, the class of δ -quasi-prototiles of monohedral tilings gets smaller. Obviously prototiles of monohedral tilings are δ -quasi-prototiles of monohedral tilings and 0-quasi-prototiles of monohedral tilings are prototiles of monohedral tilings. We make the suggestion above for considering problems because prototiles of monohedral tilings are strict and exact objects but δ -quasi-prototiles of monohedral tilings are freer relatively.

(3) (i) For a given set T , such as a tetrahedron, a pentagon, etc., determine $\delta_0 \geq 0$ so that when $\delta > \delta_0$, there exists T_δ satisfying $\tilde{h}(T_\delta, T) \leq \delta$ and T_δ is a prototile of a monohedral tiling but when $0 \leq \delta < \delta_0$ there do not exist any T_δ satisfying $\tilde{h}(T_\delta, T) \leq \delta$ such that T_δ admit monohedral tilings.

(ii) For a given set T , such as a tetrahedron, a pentagon, etc., determine $\delta_0 \geq 0$ so that T admits δ -quasi-tilings as $\delta > \delta_0$ but T admit no δ -quasi-tilings as $0 \leq \delta < \delta_0$.

(iii) Do there exist any bodies T in \mathbb{R}^n such that for some $\delta > 0$ any T_δ satisfying $\tilde{h}(T_\delta, T) \leq \delta$ admit no monohedral tilings whereas T admit monohedral δ -quasi-tilings?

(iv) Do there exist any bodies T in \mathbb{R}^n such that T admit monohedral δ -quasi-tilings for any $\delta > 0$ whereas T admit no monohedral tilings? If this kind of sets T do exist then T should be strange sets and probably possess fractal boundaries. Among these sets more strange ones, if exist, are positive answers of the following question.

(v) Do there exist any bodies T in \mathbb{R}^n such that for any $\delta > 0$ any T_δ satisfying $\tilde{h}(T_\delta, T) \leq \delta$ admit no monohedral tilings whereas T admit monohedral δ -quasi-tilings?

5.2. Quasi-patterns. For the notion of a (mono-motif) pattern (in plane) we refer to [50, Chapter 5]. We now extend this notion and as an example we also give a result, which is an extension of a basic proposition in patterns.

At first, for two families $\{M_i : i \in I\}$ and $\{N_j : j \in J\}$ (denoted $\{M_i\}$ and $\{N_j\}$) (I and J are two index sets) of sets in \mathbb{R}^n , we define

$$h(\{M_i\}, \{N_j\}) := \sup_{i \in I, j \in J} \{ \inf \{ h(M_i, N_{j'}) : j' \in J \}, \\ \inf \{ h(M_{i'}, N_j) : i' \in I \} \}.$$

5.2.1. Quasi-symmetry groups. Let $\delta \geq 0$. Let $\mathfrak{P} = \{M_i : i \in I\}$ be a family of sets in \mathbb{R}^n . A δ -(quasi-)symmetry φ of \mathfrak{P} is an isometry such that

$$h(\{\varphi(M_i)\}, \{M_i\}) \leq \delta.$$

A δ -(quasi-)symmetry group of \mathfrak{P} is a group \mathcal{G} consisting of δ -quasi-isometries of \mathfrak{P} .

5.2.2. Quasi-patterns. Let $\delta \geq 0$, $\delta_{ij} \geq 0$ and $\delta_i \geq 0$ ($i, j \in I$). Let $\mathfrak{P} = \{M_i : i \in I\}$ be a nonempty family of nonempty subsets of \mathbb{R}^n and \mathcal{G} is a δ -symmetry group. Assume

- (i) M_i ($i \in I$) are pairwise disjoint;
- (ii) Given any pairs $i, j \in I$ there exist $g_{ij} \in \mathcal{G}$ so that

$$h(M_i, g_{ij}(M_j)) \leq \delta_{ij}$$

and

$$\mathcal{G}' = \{g_{ij} : i, j \in I\}$$

is a subgroup of \mathcal{G} .

Then \mathfrak{P} is called a (*monomotif*) $\langle \delta, \{\delta_{ij}\} \rangle$ -(quasi-)pattern and each M_i ($i \in I$) is called a *motif quasi-copy* or *motif- i copy* (or (*motif*) *copy- i*) of \mathfrak{P} . A nonempty set $M \subseteq \mathbb{R}^n$ satisfying

$$\tilde{h}(M_i, M) \leq \delta_i \quad (i \in I)$$

is called a δ_i -(quasi-)motif of \mathfrak{P} . Note that \mathcal{G}' is not necessarily unique. Let \mathcal{G}_0 is a maximal element of $\{\mathcal{G}'\}$. Then we call \mathcal{G}_0 a $\langle \delta, \{\delta_{ij}\} \rangle$ -(quasi-)symmetry group of \mathfrak{P} . If $\delta_{ij} = \delta_0$ for all $i, j \in I$ then \mathfrak{P} is called a (*monomotif*) $\langle \delta, \delta_0 \rangle$ -(quasi-)pattern and a (*monomotif*) δ -(quasi-)pattern if $\delta = \delta_0$ moreover.

5.2.3. Discrete conditions. We say that a quasi-pattern $\mathfrak{P} = \{M_i : i \in I\}$ is *discrete* if the following conditions hold:

- (D1) All M_i ($i \in I$) are bounded and usually they are also connected (if all M_i are connected then \mathfrak{P} is called a *connected* quasi-pattern).
- (D2) For each $i \in I$ there is an open set E_i which contains M_i but $M_j \cap E_i = \emptyset$ for all $j \in I$ and $j \neq i$.
- (D3) The cardinal number $\#I \geq 2$.

We say that \mathfrak{P} is d_0 -discrete if (D2) is replaced by the following stronger condition:

$$(D2') \quad d_0 = \inf\{d(M_i, M_j) : i, j \in I; i \neq j\} > 0.$$

5.2.4. Engulfing and subtending. Let $\delta_1, \delta_2 \geq 0$. Let $\mathfrak{P} = \{M_i : i \in I\}$ and $\mathfrak{Q} = \{N_i : i \in I\}$ be two quasi-patterns with the same index set I .

(i) We say that $\mathfrak{Q} \langle \delta_1, \delta_2 \rangle$ -*engulfs* \mathfrak{P} if the following two conditions hold:

(E1) $N_i \supseteq M_i$ for each $i \in I$;

(E2) There exist a δ_1 -symmetry group \mathcal{G}_1 of \mathfrak{P} and a δ_2 -symmetry group \mathcal{G}_2 of \mathfrak{Q} such that $\mathcal{G}_2 \supseteq \mathcal{G}_1$.

(ii) We say that $\mathfrak{P} \langle \delta_1, \delta_2 \rangle$ -*subtends* \mathfrak{Q} if the following two conditions hold:

(S1) $M_i \subseteq N_i$ for each $i \in I$;

(S2) There exist a δ_1 -symmetry group \mathcal{G}_1 of \mathfrak{P} and a δ_2 -symmetry group \mathcal{G}_2 of \mathfrak{Q} such that $\mathcal{G}_1 \supseteq \mathcal{G}_2$.

For generality we may use the following condition to replace (E1) and (S1) above:

(ES) $\tilde{h}(M_i, N_i) \leq \delta^{(i)}$,

where $\{\delta^{(i)} : i \in I\}$ is a set of fixed nonnegative real numbers. Then we say that $\mathfrak{Q} \langle \delta_1, \delta_2, \{\delta^{(i)}\} \rangle$ -*engulfs* \mathfrak{P} and $\mathfrak{P} \langle \delta_1, \delta_2, \{\delta^{(i)}\} \rangle$ -*subtends* \mathfrak{Q} respectively.

5.2.5. Definition. (1) Let A be a nonempty set of \mathbb{R}^n . By the *infield* of A , denoted $\text{In}(A)$, we mean the complement of A_∞^c , where A_∞^c is the unbounded component of $\mathbb{R}^n \setminus A$.

(2) Let $\mathfrak{P} = \{M_i : i \in I\}$ be a quasi-pattern. If $\text{In}(M_i) \cap \text{In}(M_j) = \emptyset$ for all $i, j \in I$ and $i \neq j$, then we say \mathfrak{P} is *separated*. If each $\text{In}(M_i)$ ($i \in I$) contains a ball of radius r_0 and contained in a ball of radius R_0 , where r_0 and R_0 are two positive constants only related to \mathfrak{P} , then we say that \mathfrak{P} is *fine-distributed*.

(3) Let A be a nonempty set of \mathbb{R}^n and $\lambda \geq 0$. If λ -neighborhood $\mathcal{N}(\text{In}(A), \lambda)$ of $\text{In}(A)$ is still simply connected, then A is said to be λ -*exterior open topology-free*.

Some corresponding classical results about patterns may be considered to be extended to the case of quasi-patterns. For example, we have the following result about quasi-patterns similar to [50, 5.1.1].

5.2.6. Proposition. Let $\delta \geq 0$ and $d_0 > 4\delta$. Suppose $\mathfrak{P} = \{M_i : i \in I\}$ is a d_0 -discrete connected δ -pattern in \mathbb{R}^2 with a δ -symmetry group \mathcal{G}_1 . Then \mathfrak{P} can be $\langle \delta, 4\delta + 2\varepsilon_0 \rangle$ -engulfed by a separated and fine-distributed connected $\langle 4\delta + 2\varepsilon_0, 0 \rangle$ -pattern \mathfrak{Q} (ε_0 is any fixed number satisfying $0 < \varepsilon_0 < \frac{d_0 - 4\delta}{2}$). If there is a motif quasi-copy $M_{i_0} \in \mathfrak{P}$ which is λ -exterior open topology-free, where $\lambda = 5\delta + 3\varepsilon_0 + \varepsilon$ and ε is any sufficiently small positive number, then \mathfrak{P}

can be $\langle \delta, 4\delta + 2\varepsilon_0 \rangle$ -engulfed by a fine-distributed open disk $\langle 4\delta + 2\varepsilon_0, 0 \rangle$ -pattern \mathfrak{Q} (all motif quasi-copies of \mathfrak{Q} are topological disks).

Proof. Take $r = \delta + \varepsilon_0$, where $0 < \varepsilon_0 < \frac{d_0 - 4\delta}{2}$. Suppose

$$N_{i_0} := \mathcal{N}_r(M_{i_0}).$$

Let $g_{ji_0} \in \mathcal{G}_1$ satisfy

$$h(g_{ji_0}(M_{i_0}), M_j) \leq \delta \quad (5.1)$$

for $j \in I$ and $j \neq i_0$. Suppose

$$N_j := g_{ji_0}(N_{i_0}). \quad (5.2)$$

Then each N_j ($j \in I$) is connected. Below we show that

$$\mathfrak{Q} := \{N_i : i \in I\}$$

is a separated fine-distributed $\langle 4\delta + 2\varepsilon_0, 0 \rangle$ -pattern and \mathfrak{Q} $\langle \delta, 4\delta + 2\varepsilon_0 \rangle$ -engulfs \mathfrak{P} .

First, we prove $N_i \cap N_j = \emptyset$ for $i, j \in I$ and $i \neq j$. Let $0 < \varepsilon < \frac{d_0 - 4\delta - 2\varepsilon_0}{3}$. Select suitable $n_i \in N_i$, $n_j \in N_j$, $n'_{i_0}, n''_{i_0} \in N_{i_0}$, $m'_{i_0}, m''_{i_0} \in M_{i_0}$, $m_i \in M_i$ and $m_j \in M_j$ so that

$$\begin{aligned} d(N_i, N_j) &> d(n_i, n_j) - \varepsilon \\ &\geq d(m_i, m_j) - d(n_i, m_i) - d(n_j, m_j) - \varepsilon \\ &\geq d(M_i, M_j) - [d(g_{ii_0}(n'_{i_0}), g_{ii_0}(m'_{i_0})) + d(g_{ii_0}(m'_{i_0}), m_i)] \\ &\quad - [d(g_{ji_0}(n''_{i_0}), g_{ji_0}(m''_{i_0})) + d(g_{ji_0}(m''_{i_0}), m_j)] - \varepsilon \\ &\geq d_0 - 2[r + (\delta + \varepsilon)] - \varepsilon = d_0 - 4\delta - 2\varepsilon_0 - 3\varepsilon > 0. \end{aligned}$$

For $i \in I$, $g \in \mathcal{G}_1$ and $\varepsilon > 0$, since $gg_{ii_0} \in \mathcal{G}_1$, we can take a suitable $j \in I$ such that

$$h(gg_{ii_0}(M_{i_0}), M_j) < \delta + \varepsilon.$$

Thus

$$\begin{aligned} h(g(N_i), N_j) &\leq h(gg_{ii_0}(N_{i_0}), gg_{ii_0}(M_{i_0})) + h(gg_{ii_0}(M_{i_0}), M_j) \\ &\quad + h(M_j, g_{ji_0}(M_{i_0})) + h(g_{ji_0}(M_{i_0}), g_{ji_0}(N_{i_0})) \\ &< r + (\delta + \varepsilon) + \delta + r = 4\delta + 2\varepsilon_0 + \varepsilon. \end{aligned}$$

By the same reasoning we can also get

$$h(N_i, g(N_j)) < 4\delta + 2\varepsilon_0 + \varepsilon$$

for a suitable $j \in I$. Therefore we have

$$h(\{g(N_i)\}, \{N_i\}) \leq 4\delta + 2\varepsilon_0. \quad (5.3)$$

For $i, j \in I$, let $g = g_{ji_0}g_{ii_0}^{-1}$. Then $g \in \mathcal{G}_1$ and

$$g(N_i) = g_{ji_0}g_{ii_0}^{-1}g_{ii_0}(N_{i_0}) = g_{ji_0}(N_{i_0}) = N_j.$$

Consequently

$$h(g(N_i), N_j) = 0.$$

It is obvious that \mathfrak{Q} is fine-distributed. Since N_i ($i \in I$) are disjoint from and isometric to each other, it follows that the bounded components $C_{k_i}^{(i)}$ ($k_i = 1, 2, \dots$) of the complement of N_i are disjoint from each N_j and from the bounded components $C_{k_j}^{(j)}$ ($k_j = 1, 2, \dots$) of the complement of N_i ($j \in I, j \neq i$). Hence \mathfrak{Q} is separated.

From (5.1) and (5.2) it follows that $M_j \subseteq N_j$ for all $j \in I$. We may choose \mathcal{G}_2 to be a maximal element of $\{\mathcal{G}'\}$ satisfying the definition of $\langle 4\delta + 2\varepsilon_0, 0 \rangle$ -patterns according to 5.2.2 and including \mathcal{G}_1 .

Finally let M_{i_0} be λ -exterior open topology free for $\lambda = 5\delta + 3\varepsilon_0 + \varepsilon$, where ε is any sufficiently small positive number, and let $N_i^* = \text{In}(N_i)$ ($i \in I$). Then by (5.3) we can obtain

$$h(\{g(N_i^*)\}, \{N_i^*\}) \leq 4\delta + 2\varepsilon_0.$$

Therefore $\mathfrak{Q}^* = \{N_i^* : i \in I\}$ is an open disk $\langle 4\delta + 2\varepsilon_0, 0 \rangle$ -pattern. \square

5.2.7. Corollary. *Let $\delta \geq 0$ and $d_0 > 4\delta$. Suppose \mathfrak{P} is a d_0 -discrete connected δ -pattern in \mathbb{R}^2 . Then \mathfrak{P} is locally finite (i.e., set $\{M : M \in \mathfrak{P}, M \cap D \neq \emptyset\}$ is finite for any disk D).*

5.3. Quasi-packings.

5.3.1. Packing density. We recall that a family $\mathbf{P} = \{P_i : i \in I\}$ (I is an at most countable index set) of compact sets with nonempty interiors is said to form a *packing* in a domain $\Omega \subseteq \mathbb{R}^n$ if $\bigcup_{i \in I} P_i \subseteq \Omega$ and no two members of \mathbf{P} have an interior point in common. Suppose a bounded domain D and members P_i ($i \in I$) of \mathbf{P} are Jordan-measurable. The density of the packing \mathbf{P} relative to D is defined as

$$\mathbf{d}(\mathbf{P}, D) := \frac{\text{Vol}(P_i \cap D)}{\text{Vol}(D)}.$$

The *upper* and *lower densities* of \mathbf{P} (in \mathbb{R}^n) are

$$\overline{\mathbf{d}}(\mathbf{P}) := \limsup_{r \rightarrow +\infty} \mathbf{d}(\mathbf{P}, B(r))$$

and

$$\underline{\mathbf{d}}(\mathbf{P}) := \liminf_{r \rightarrow +\infty} \mathbf{d}(\mathbf{P}, B(r))$$

respectively, where $B(r) := \{x \in \mathbb{R}^n : d(x, o) < r\}$ (o denotes the origin). If these two numbers are the same, then it is called the *density* of \mathbf{P} (in \mathbb{R}^n), denoted $\mathbf{d}(\mathbf{P})$. Let P be an n -dimensional compact set with nonempty interior. Then the *packing density* of P is the largest density of a packing of congruent copies of P in \mathbb{R}^n .

For the notion of packings we refer to [19, 1.1] and [40], etc.

5.3.2. Example. Let $\varepsilon \geq 0$. Let P be the dodecagon $A_0A_1 \cdots A_{11}$, where the vertexes are: $A_0(0, 0)$, $A_1(2, 0)$, $A_2(2, 4)$, $A_3(3, 4)$, $A_4(3, 0)$, $A_5(5, 0)$, $A_6(5, 5)$, $A_7(3 + \varepsilon, 5)$, $A_8(3 + \varepsilon, 8)$, $A_9(2 - \varepsilon, 8)$, $A_{10}(2 - \varepsilon, 5)$, $A_{11}(0, 5)$. We consider a packing of congruent copies of P in the plane in two cases: (i) $\varepsilon > 0$ and (ii) $\varepsilon = 0$. We will find the packings in the two cases have an obvious difference, even if ε is sufficiently small in case (i). The packing densities have an obvious jump even if the difference of ε and 0 is very very tiny.

This example, along with experience of life, tells us that sometimes if we “squeeze” bodies we may obtain much more dense packings. Thus we may give a corresponding notion below.

5.3.3. Quasi-packing density. Let $P \subseteq \mathbb{R}^n$ be a compact set with nonempty interior and $\delta \geq 0$. If $Q \subseteq \mathbb{R}^n$ satisfies

$$\tilde{h}(Q, P) \leq \delta$$

then Q is called a δ -(*quasi*)-copy of P .

We define the δ -(*quasi*)-packing density $\mathbf{d}_\delta(P)$ of P as the largest density of a packing of δ -copies with nonempty interiors of P in \mathbb{R}^n (this packing is called a δ -(*quasi*)-packing of P (in \mathbb{R}^n)), i.e.,

$$\mathbf{d}_\delta(P) := \sup\{\mathbf{d}(\mathbf{P}) : \mathbf{P} \text{ is a } \delta\text{-packing of } P \text{ in } \mathbb{R}^n\}.$$

If $\mathbf{d}_\delta(P)$ as a function of δ is not continuous at $\delta = \delta_0$, then P is said to possess the *collapse property* at $\delta = \delta_0$, δ_0 is called a *collapse value* of P and

$$\Delta_{\delta_0}(P) := \lim_{\delta \rightarrow \delta_0^+} \mathbf{d}_\delta(P) - \lim_{\delta \rightarrow \delta_0^-} \mathbf{d}_\delta(P)$$

is called the *collapse quantity* of P at $\delta = \delta_0$.

In Example 5.3.2 it is easy to see that P possesses the collapse property at $\delta = \frac{\varepsilon}{2}$.

5.3.4. Some related questions. (1) Give a few more “natural” examples than Example 5.3.2 (we may see some “natural” strange phenomena in [112]).

(2) How much is $\sup\{\Delta_{\delta_0}(P) : P \text{ is a compact connected set with nonempty interior in } \mathbb{R}^n \text{ which possesses the collapse property at } \delta = \delta_0\}$?

(3) Do there exist any compact connected sets P with nonempty interiors in \mathbb{R}^n which have infinitely many collapse values with 0 as an accumulation point? If the answer is positive, then may all these collapse values of some P have an uncountable cardinal?

5.4. Approximate crystals. For nearly two hundred years the internal structure of a crystal was considered periodic. The discovery of X-ray diffraction and Laue's experiment (by W. Friedrich and P. Knipping) supported this viewpoint. A (classical) crystal may be defined as the union of a finite number of orbits of a crystallographic group. In this subsection we refer to [97]) and [98].

After the announcement of the discovery of crystals with icosahedral symmetry in the year 1984 the classical viewpoint has been extended. In 1992 the Commission on Aperiodic Crystals of the International Union of Crystallography proposed as a working definition: a *crystal* is a solid with an essentially discrete diffraction pattern. In [98] a (*generalized*) *crystal* is defined as a Delone set Λ with nontrivial Λ_d . Here a *Delone set* is an (r, R) *system* ($r, R > 0$), i.e., a set Λ of points in \mathbb{R}^n that is r -discrete and *relatively dense* (every sphere of radius R contains at least one point of Λ).

On the other hand, in the case of noncrystals, because of their disorder, the method of radial distribution functions is applied to deal with this situation.

Below let us try to consider the problem from another way.

5.4.1. Approximate crystals. Let $\delta \geq 0$. A δ -*approximate crystal* (δ -*crystal*) is defined as a δ -*dot-pattern* (δ -pattern consisting of singletons) with a crystallographic group as its δ -symmetry group (called a δ -*symmetry crystallographic group*).

Let $\mathfrak{P} = \{P_i : i \in I\}$ be a δ -crystal and let \mathfrak{G} denote the set of all its δ -symmetry crystallographic groups. Let

$$\lambda(\mathcal{G}) := \frac{\inf\{h(\{P_i\}, \{g(P) : g \in \mathcal{G}\}) : P \in \mathbb{R}^n\}}{d_{\mathcal{G}}},$$

where

$$d_{\mathcal{G}} := \inf\{d(g_1(P), g_2(P)) : g_1, g_2 \in \mathcal{G}, P \in \mathbb{R}^n, g_1(P) \neq g_2(P)\},$$

for $\mathcal{G} \in \mathfrak{G}$. It is easy to see $d_{\mathcal{G}} \neq 0$. Define

$$\lambda = \lambda_{\mathfrak{G}} := \inf\{\lambda(\mathcal{G}) : \mathcal{G} \in \mathfrak{G}\}. \quad (5.4)$$

We call \mathfrak{P} a λ -*approximate crystal* or λ -*crystal*.

If there exists $\mathcal{G} \in \mathfrak{G}$ and $P \in \mathbb{R}^n$ such that for each $i \in I$ there exists $g \in \mathcal{G}$ satisfying

$$d(P_i, g(P)) = \lambda_{\mathfrak{G}} d_{\mathcal{G}}, \quad (5.5)$$

then \mathfrak{P} is called a *strict $\lambda \sim$ crystal*.

Now we give simple and obvious definitions of an approximate crystal in two ways.

(1) *A quasi-tiling model.* Let $\mathcal{T} = \{T_i : i \in I\}$ be a monohedral tiling model of a crystal \mathbf{C} with a prototile P_0 and $\delta \geq 0$. Assume $\mathcal{T}_\delta = \{T'_i : i \in I\}$ is a quasi-tiling whose tiles are still polyhedra obtained by perturbing \mathcal{T} such that

$$\tilde{h}(T'_i, P_0) \leq \delta r(P_0), \quad (5.6)$$

where $r(P_0)$ is the radius of P_0 (see Definition 2.6.2). If a non-crystal \mathbf{C}_δ is obtained by perturbing the crystal \mathbf{C} and \mathbf{C}_δ possesses tiling model \mathcal{T}_δ , then \mathbf{C}_δ is called a *$\delta \sim \mathbf{C}$ -crystal*.

(2) *A discrete point-set model.* Let $\mathfrak{C} = \{C_i : i \in I\}$ be a dot-pattern model of a normal crystal and $\lambda \geq 0$. Let $\mathfrak{P} = \{P_i : i \in I\}$ be a discrete set of points. Suppose

$$\tilde{h}(\mathfrak{P}, \mathfrak{C}) \leq \lambda d_{\mathfrak{C}}, \quad (5.7)$$

where

$$d_{\mathfrak{C}} = \min\{d(C_i, C_j) : i, j \in I, i \neq j\}. \quad (5.8)$$

Then \mathfrak{P} is called a *$\lambda \sim \mathfrak{C}$ -crystal*. If moreover the equality in (5.7) always holds, i.e.,

$$\tilde{h}(\mathfrak{P}, \mathfrak{C}) = \lambda d_{\mathfrak{C}}, \quad (5.9)$$

then \mathfrak{P} is called a *strict $\lambda \sim \mathfrak{C}$ -crystal*.

5.4.2. Some related questions. Given a substance \mathbf{S} ignoring its internal structure, let Σ be the set of all approximate crystals consisting of the same substance \mathbf{S} . Suppose \mathfrak{C} is the crystal consisting of \mathbf{S} . Since a $0 \sim \mathfrak{C}$ -crystal is a normal crystal, it possesses a discrete diffraction pattern. We may imagine that when $\lambda > 0$ is sufficiently small a $\lambda \sim \mathfrak{C}$ -crystal may still produces a diffraction pattern. Now we ask the following questions.

(1) How much is $\lambda_{\mathbf{S}} := \sup\{\lambda : \mathfrak{P} \in \Sigma \text{ is a (well-distributed random strict) } \delta \sim \mathfrak{C}\text{-crystal, where } 0 \leq \delta \leq \lambda, \text{ which produces an essentially discrete diffraction pattern}\}$?

(2) For different kinds of substances \mathbf{S} , are $\lambda_{\mathbf{S}}$ the same or different?

(3) Furthermore, find the laws of changes of the properties (in mechanics, acoustics, heat, optics, electricity and magnetism, etc.) of a (well-distributed random strict) $\lambda \sim \mathfrak{C}$ -crystal depending on λ .

5.4.3. Remark. (1) In the research if we weaken the demand of “strict” in the strict $\lambda \sim$ crystal or strict $\lambda \sim \mathfrak{C}$ -crystal, e.g. we require “strict” in nanometer or over scales but ignore “strict” in angstrom scales (for nanometer solid materials) the problem perhaps becomes easier.

(2) Since the scanning tunneling microscope (STM) was invented, Richard P. Feynman's imagination that we could arrange the atoms one by one the way we want them has seen the dawn of its realization. The questions that we ask above are just some more concrete ones related to Feynman's question: *what would the properties of materials be if we could really arrange the atoms the way we want them?* (see [43])

(3) Experts in different research fields may put forward different related questions. For example, one may pose similar questions for substances in liquid or gaseous states or for organic compounds, etc.

(4) In fact we could ask a few of questions in this subsection, some of which might seem abrupt or even unreasonable. For example, we may ask the question: if we had calculated out the critical values (interval) of the shape vision error (see Section 4) and $\lambda_{\mathcal{S}}$, what might we say about them?

(5) As examples, we define several concepts for further consideration. Let $\mu > 0$, $\lambda \geq 0$ and $0 \leq \lambda_1 \leq \lambda_2$.

(i) Let $\mathfrak{C} = \{C_i : i \in I\}$ be a dot-pattern model of a normal crystal. Let $\mathfrak{P} = \{P_i : i \in I\}$ be a discrete set of points. Suppose (5.7) or (5.9) is now changed into

$$\lambda_1 d_{\mathfrak{C}} \leq \tilde{h}(\mathfrak{P}, \mathfrak{C}) \leq \lambda_2 d_{\mathfrak{C}}, \quad (5.10)$$

where $d_{\mathfrak{C}}$ has been defined in (5.8). Then \mathfrak{P} is called a $\langle \lambda_1, \lambda_2 \rangle \sim \mathfrak{C}$ -crystal.

(ii) Let Λ be a *uniformly discrete* set of points in \mathbb{R}^n , i.e.,

$$d_{\Lambda} := \inf\{d(P_1, P_2) : P_1, P_2 \in \Lambda\} > 0.$$

If

$$\tilde{h}(\mu\Lambda, \Lambda) \leq \lambda d_{\Lambda},$$

then Λ is said to possess $\langle \mu; \lambda \rangle$ (-approximate) *inflation symmetry* (cf. [98]).

(iii) If in (i) we change (5.10) into

$$\lambda_1 d_{\mathfrak{C}} \leq \tilde{h}(\mu\mathfrak{P}, \mathfrak{C}) \leq \lambda_2 d_{\mathfrak{C}},$$

then \mathfrak{P} is called a $\langle \mu; \lambda_1, \lambda_2 \rangle \sim \mathfrak{C}$ -crystal. Similarly we may change (5.5) into

$$\left| \frac{d(P_i, g(P))}{d_{\mathcal{G}}} - \lambda_{\mathfrak{G}} \right| \leq \delta,$$

where $\delta \geq 0$ and $\lambda_{\mathfrak{G}}$ is the one defined in (5.4), and change (5.6) into

$$\delta_1 r(P_0) \leq \tilde{h}(T'_i, P_0) \leq \delta_2 r(P_0)$$

or more generally

$$\delta_1 r(P_0) \leq \tilde{h}(\mu T'_i, P_0) \leq \delta_2 r(P_0),$$

where $0 \leq \delta_1 \leq \delta_2$, to obtain corresponding definitions. In the same manner we may give more general notions than the ones in quasi-tilings and quasi-patterns.

(6) In this paper if metric \tilde{h} is replaced by \bar{h} and \hat{h} is replaced by \check{h} , corresponding concepts can be defined and corresponding results and questions can be considered similarly.

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